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$\aleph_0$-Categoricity of Groups

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A countable group is $\aleph_0$-categorical if it can be characterized, up to isomorphism, within the class of countable groups, by its first-order properties. In this paper we discuss various kinds of $\aleph_0$-categorical groups. There are five main sections—on Abelian groups, on direct sums of finite groups, on groups with large Abelian subgroups, on certain direct limits of finite groups, and on Burnside groups—and an introduction intended to explicate the first sentence of this paper and to lay the logical groundwork for what follows.

The following paragraphs convey the sort of results we have obtained in the respective sections.

An Abelian group is $\aleph_0$-categorical if and only if it is a group of bounded order.

Let G be a direct sum of copies of the finite groups $G_1, G_2, \ldots, G_n$. Then G is $\aleph_0$-categorical if and only if every G_i which occurs infinitely often (in the direct sum) is Abelian.

Let G be an infinite group with a normal Abelian subgroup H of exponent n and index q; such a group is called an $n - q$ group. Every $n - q$ group, where n is square-free and q is prime, is $\aleph_0$-categorical. In proving this theorem, we also prove structure theorems for such groups.

Let H be a finite group. Then a certain direct limit of direct sums of copies of H is $\aleph_0$-categorical.

Let $B(r, n)$ be the Burnside group of exponent n on r generators—where r is allowed to take on the value $\aleph_0$. If the Burnside conjecture is false for n, i.e., if for some r_0, $B(r_0, n)$ is infinite, then for all r, $r_0 \leqslant r \leqslant \aleph_0$, $B(r, n)$ is not $\aleph_0$-categorical. Furthermore, the Burnside group $B(\aleph_0, p)$ is not $\aleph_0$-categorical, for any odd prime p.

As can be seen from the theorems above (and even more from their proofs) the determination of whether or not a particular group is $\aleph_0$-categorical is basically an algebraic, rather than a logical, problem. The class of $\aleph_0$-categorical groups thus seems to be an object of algebraic interest; the main question to be answered is whether or not this class can be characterized algebraically (as we have done, e.g., with the $\aleph_0$-categorical Abelian groups). This paper is an initial attempt to shed some light on this question.

Logicians have long been interested in $\aleph_0$-categorical structures from a model-theoretic point of view. The main theorem in the subject was proved (independently) by Engeler (E. Engeler, A characterization of theories with isomor-

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phic denumerable models, *Amer. Math. Soc. Notices* **6** (1959), 161), Ryll-Nardzewski (C. Ryll-Nardzewski, On the categoricity in power $\leqslant \aleph_0$, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys.* **7** (1959), 545–548), and Svenonius (L. Svenonius, $\aleph_0$-categoricity in first-order predicate calculus, *Theoria (Lund)* **25**, (1959), 82–94). Further investigations have been carried out by Waskiewicz and Weglorz (J. Waskiewicz and B. Weglorz, On $\aleph_0$-categoricity of powers, *Acad. Polon. Sci. Ser. Sci. Math., Astron. Phys.* **17** (1969), 195–199), Rosenstein (J. G. Rosenstein, $\aleph_0$-categoricity of linear orderings, *Fund. Math.* **44** (1969), 1–5), Glassmire (W. Glassmire, Jr., A Problem in Categoricity, *Amer. Math. Soc. Notices* **17** (1970), 295), and Ash (C. J. Ash, $\aleph_0$-categorical theories, to appear). Nerode and Crossley (A. Nerode and J. N. Crossley, Effective Dedekind Types, in preparation) have recently observed that the work of Dekker, Myhill, and Nerode on recursive equivalence types can be generalized to arbitrary $\aleph_0$-categorical structures. Theorem 2, which was proved in 1968, has been used by Plotkin (J. Plotkin, Generic Embeddings, *J. Symbolic Logic* **34** (1969), 388–394); it has also been proved by Macintyre (A. Macintyre, Categoricity in power for some algebraic theories, *J. Symbolic Logic* **35** (1970), 606) and Eklof and Fisher (P. Eklof and E. R. Fisher, The Elementary Theory of Abelian Groups, to appear).

It is intended that both the logician and the group theorist should find this paper essentially self-contained.

1. Introduction

A first order property is a property which can be formulated within the first order predicate calculus, i.e. a property which can be written as a finite expression involving only the group operations, the logical connectives (and, or, not, if $\cdots$ then $\cdots$, iff), and quantifiers which range over elements (!) of the group. Thus commutativity is a first order property $(\forall x)(\forall y)\ (x \cdot y = y \cdot x)$, whereas simplicity is apparently not since it involves the existence of a normal subgroup and hence is formulated in terms of a so-called second-order quantifier which ranges over *sets* of elements of the group. [Note the word "apparently." Just because the usual definition of a certain property involves second-order quantifiers does not mean that there is no first order reformulation of the property. For example, semisimplicity of a ring involves intersecting maximal ideals (second-order!) but can be reformulated in terms of elements (first order) as follows: $(\forall x)(\exists y)(x + y + xy = 0)$.]

It might seem, at first glance, that a presentation of a group is in effect a list of first order properties of the group. For a presentation consists of a set of generators and a set of words on these generators (called defining relations), and an understanding that every element of the group can be expressed as a product of generators and that every relation of the group (i.e. every word on the generators which equals the identity) can be expressed as a product of defining relations and their conjugates; all of which appears to be first order.

However such a presentation cannot be easily expressed in the first order predicate calculus. For to say that "a group is generated by a single element" is to say that "there is an element x such that any element y can be expressed as a power of x," which can be written

$$(\exists x)(\forall y)(\exists n)(y = x^n)$$

or

$$(\exists x)(\forall y)(\cdots \vee y = x^{-1} \vee y = x^0 \vee y = x^1 \vee y = x^2 \vee \cdots)$$

neither of which are first order expressions (since, in the first, one of the quantifiers ranges over the integers instead of over the elements in the group, and, in the second, an infinite disjunction occurs.) [Note the word "easily." We have not yet proved that the property of being cyclic is not a first order property; rather we have shown that the usual definition involves non-first-order concepts. That "cyclicness" is, in fact, not a first order property follows from Theorem 1.] The difficulty above, of course, results from the fact that the phrase "a product of" contains a hidden numerical quantifier. If the cyclic group is finite, say of exponent n, then we can replace the quantifier by a finite disjunction, viz.

$$(\exists x)(\forall y)(y = e \vee y = x \vee y = x^2 \vee y = x^3 \vee \cdots \vee y = x^{n-1}).$$

If it is infinite, we have no such opportunity.

We say that two groups are *first order equivalent (elementarily equivalent)* if they have precisely the same first order properties. A countable group G is $\aleph_0$-categorical if any countable group which is elementarily equivalent to G is isomorphic to G, so that G is "characterized up to isomorphism, within the class of countable groups, by its first order properties." For the purposes of this paper we shall assume that countable means finite or countably infinite; note that no infinite group is elementarily equivalent to any finite group, for any finite group has one of the first order properties $(\exists x_1) \cdots (\exists x_n)(\forall y)[y = x_1 \vee y = x_2 \vee \cdots \vee y = x_n]$ whereas any infinite group has none of them; note also that any finite group is $\aleph_0$-categorical since we can, so to speak, transcribe its multiplication table into a first order statement. [For example, for $Z_2 \times Z_2$ we can write

$$\begin{aligned}(\exists x)(\exists y)(\exists z)(\exists w)\{&x \neq y \wedge x \neq z \wedge x \neq w \wedge y \neq z \wedge y \neq w \wedge z \neq w \\ &\wedge (\forall v)[v = x \vee v = y \vee v = z \vee v = w] \\ &\wedge (x \cdot x = x) \wedge (x \cdot y = y) \wedge (x \cdot z = z) \wedge (x \cdot w = w) \\ &\wedge (y \cdot x = y) \wedge (y \cdot y = x) \wedge (y \cdot z = w) \wedge (y \cdot w = z) \\ &\wedge (z \cdot x = z) \wedge (z \cdot y = w) \wedge (z \cdot z = x) \wedge (z \cdot w = y) \\ &\wedge (w \cdot x = w) \wedge (w \cdot y = z) \wedge (w \cdot z = y) \wedge (w \cdot w = x)\}.]\end{aligned}$$

The discussion of the preceding paragraphs leads us to the supposition that although any group can be defined by means of a presentation, not every group can be defined by first order properties. From this point of view, our purpose in this paper is to determine which groups can be defined by their first order properties.

Just as certain expressions were considered above as first order properties of groups, so other expressions (involving the same symbols) can be thought of as first order properties of elements—and ordered n-tuples of elements—of groups. For example, the expression $(\forall w)(v_1 w = w v_1)$ says that "v_1 is the center," $(\exists w)(v_1 = w^{-1} v_2 w)$ says that "v_1 is conjugate to v_2," $(\exists x)(\exists y)(v_1 = x^{-1}y^{-1}xy)$ says that "v_1 is a commutator," etc. The totality of such expressions, involving only the variables $v_1, v_2, ..., v_n$ unquantified will be denoted P^n, and can be thought of as the set of all first order properties that an n-tuple of elements of a group may have.

For each group G and any positive number n we define an equivalence relation on G^n (the set of ordered n-tuples of elements of G) by stipulating that if $a, b \in G^n$ then *a is logically equivalent to b* if they have precisely the same first order properties as n-tuples of elements of G; we will write this $a \equiv_{G,n} b$, or simply $a \equiv b$ if there is no danger of confusion.

The main logical tool of this paper is a theorem due (independently) to Engeler [4], Ryll-Nardzewski [15] and Svenonius [16] which states in effect that G is $\aleph_0$-categorical if and only if $G^n/\equiv_{G,n}$ is finite for each n. Thus to show that G is not $\aleph_0$-categorical it suffices to find, for some n, an infinite list $\{d_i \mid i \in N\}$ of distinct elements of G^n and an infinite list $\{\phi^j \mid j \in N\}$ of distinct first order properties in P^n such that d_i has property ϕ^j in G if and only if $i = j$. (We shall refer to this criterion as (#).)

Applying this result we can prove the following theorem about $\aleph_0$-categorical groups.

THEOREM 1. *Let G be an $\aleph_0$-categorical group. Then G is of bounded order, i.e., there is an n such that $g^n = 1$ for every $g \in G$.*

Proof. We first show that if G has an element g of infinite order then G is not $\aleph_0$-categorical. For let $\phi^j(v_1, v_2)$ be the first order property $v_2 = v_1{}^j$ for each $j \in N$ and let a_i be $\langle g, g^i \rangle$ for each $i \in N$. But if g has infinite order then clearly a_i has property ϕ^j iff $i = j$, so that G is not $\aleph_0$-categorical. Hence every element of G has finite order. If these orders are unbounded then we can find an increasing sequence $n_0, n_1, n_2, ...,$ of natural numbers and a sequence $g_0, g_1, g_2, ...,$ of elements of G such that g_i has order n_i for each i. But then we have a sequence $\{g_i \mid i \in N\}$ of distinct elements of G and a sequence $\{v_1^{n_j} = 1 \mid j \in N\}$ of distinct first order properties such that g_i has property ϕ^j iff $i = j$, so that again G is not $\aleph_0$-categorical. Hence there is a number r such that every element of G has order at most r. By taking the least common multiple of the orders of elements of G we can find an n such that $g^n = 1$ for every $g \in G$. ▮

DEFINITION. The least n such that $g^n = 1$ for all $g \in G$, if such exists, will be called the *exponent of* G.

To show that a group G is $\aleph_0$-categorical it suffices to find a list T of first order properties which G has and which G shares with no different (i.e., nonisomorphic) countable group; for then any countable group which has *all* the first order properties of G certainly has the properties of T and thus is isomorphic to G. This set T can be thought of as a set of axioms for G, or as a first order definition of G, and is intrinsically of logical interest.

It is possible to give another, purely algebraic, necessary and sufficient condition for a group to be $\aleph_0$-categorical.

For each group G and any positive integer n, we define another equivalence relation on G^n by stipulating that if $a = \langle a_1, ..., a_n \rangle$ and $b = \langle b_1, ..., b_n \rangle$ are elements of G^n then a and b are *automorphically equivalent in* G if there is an automorphism of G which sends each a_i to b_i; we will write this as $a \sim_{G,n} b$, or simply $a \sim b$ if there is no danger of confusion. It is possible to show that if $a \sim b$ then $a \equiv b$; the proof of this intuitively clear fact requires, however, a more precise (hence more technical) definition of the notion of first order property and so (since the logician will have seen it and the algebraist will not want to see it) we omit the proof. The converse is false in general.

Thus to show that G is $\aleph_0$-categorical it suffices to show that for each n there are a finite number $k(n)$ of elements $a^1, a^2, ..., a^{k(n)}$ of G^n such that any element $b \in G^n$ is automorphically equivalent to one of $a^1, a^2, ..., a^{k(n)}$. On the other hand it is possible to show (see Vaught [17]) that if G is $\aleph_0$-categorical then $a \equiv b$ implies $a \sim b$, so that for each n the number of equivalence classes modulo $\sim_n$ is finite. (It should be noted however that this criterion is by no means necessarily easier, in an absolute sense, to apply, and that it just gives a determination as to whether or not the group is $\aleph_0$-categorical, with no indication as to how to find a set of axioms for G.) We shall make little explicit use of this criterion, but the algebraically oriented reader can use it to give different proofs of some of our theorems.

The reader should be cautioned in one regard. It is not possible to expect the set of first order properties of a group to characterize it, up to isomorphism, within the class of *all* groups. For the Lowenheim–Skolem Theorem implies that given any infinite group there are uncountable groups which are indistinguishable from it in the first order predicate calculus. Those who wish to pursue this matter may refer to [2] or [9].

We wish to call the reader's attention at this point to several abbreviations and conventions we shall adopt in this paper. If I is a finite set, and for each $i \in I$, ϕ_i is an expression of the first order predicate calculus, then we shall write $\bigwedge_{i \in I} \phi_i$ for the conjunction of the ϕ_i's and $\bigvee_{i \in I} \phi_i$ for the disjunction of the ϕ_i's. If $\phi(x, x_1, \ldots, x_k)$ is an expression of the first order predicate calculus in which the variable x is free we shall abbreviate

$$(\exists y_1)(\exists y_2) \cdots (\exists y_n) \left[\bigwedge_{1 \leqslant i \leqslant n} \phi(y_i, x_1, \ldots, x_n) \wedge \bigwedge_{1 \leqslant i < j \leqslant n} y_i \neq y_j \right]$$

to $(\exists \geqslant^n x)\, \phi(x, x_1, \ldots, x_k)$ and we shall use $(\exists!^n x)\, \phi(x, x_1, \ldots, x_k)$ instead of $(\exists \geqslant^n x)\, \phi(x, x_1, \ldots, x_k) \wedge (\sim(\exists \geqslant^{n+1} x)\, \phi(x, x_1, \ldots, x_k))$.

A set of statements which is consistent will sometimes be called *a theory*. If G is a group and T is a theory we will say that *G is a model of T*, written $G \models T$, if each of the statements in T is in fact a property of G. If $\phi \in P^n$ and $a = \langle a_1, \ldots, a_n \rangle \in G^n$ then $G \models \phi[a_1, a_2, \ldots, a_n]$ will be used if a has property ϕ in G. [All of these notions can be made excruciatingly precise, and the reader who wishes to pursue these notions can refer to [2] or [9].]

Since we will be dealing only with countable groups we will henceforth assume that all groups are countable.

In presenting a group we will use the notation $\{\cdots ; \cdots\}$. The symbols to the left of the semicolon will be the generators of the group and the equations to the right will be the defining relations of the group.

2. Abelian Groups

If a group which is of bounded order happens also to be Abelian, then its structure is easily determined. In fact an Abelian group of bounded order is a direct sum of cyclic groups whose orders are powers of primes (see Kaplansky [7, p. 17]). Using this information we can prove the following converse to Theorem 1.

Theorem 2. *Any Abelian group M of bounded order is $\aleph_0$-categorical.*

Proof. Since M is Abelian, we shall take the liberty of switching to additive notation. Let n be such that $na = 0$ for all $a \in M$. Write M as a direct sum of cyclic groups of prime power order; each of the summands has order $\leqslant n$. Assume that for each $t \leqslant n$ there are exactly s_t summands of order t. Thus $s_t = 0$ if t is not a prime power and $s_t = \omega$ if there are infinitely many summands of order t. Thus $M = \sum^{\oplus}_{2 \leqslant t \leqslant n} M_t$, where each $M_t = \sum^{\oplus}_{0 \leqslant j < s_t} M_t^j$ and each M_t^j is a cyclic group of order t.

We shall define a set T_M of statements such that $M \models T_M$ and such that if $N \models T_M$ then $M \simeq N$. The set T_M of statements in our original manuscript expressed, in a first-order way, that, for each t, M had exactly s_t summands of order t. The referee observed that the proof could be somewhat shortened if the $\{s_t \mid t > 1\}$ were viewed instead as the Ulm invariants.

Thus, following Kaplansky [7, p. 27], we define

$$\begin{aligned} M_{p,k} &= p^k M \text{ for each prime } p \text{ and } k \geqslant 1; \\ P_p &= \{x \mid px = 0\} \text{ for each prime } p; \\ P_{p,k} &= P_p \cap M_{p,k} \text{ for each prime } p \text{ and } k \geqslant 1; \end{aligned}$$

and

$$f_p(k-1) = \dim(P_{p,k-1}/P_{p,k}) \text{ for each prime } p \text{ and } k \geqslant 1.$$

Ulm's Theorem, together with the fact that an Abelian torsion group is a direct sum of primary groups, implies that an Abelian group of bounded order is completely determined by the Ulm invariants $\{f_p(k-1) \mid p \text{ prime}, k \geqslant 1\}$.

Now if $t = p^k$ then $s_t = f_p(k-1)$ so that, instead of saying that M has exactly s_t summands of order t, it suffices to say that the dimension of $P_{p,k-1}/P_{p,k}$ is s_t. But, for any $r > 0$, the statement

$$\begin{aligned} \Phi(p, r)\colon (\exists x_2)(\exists x_3) \cdots (\exists x_{p^r}) \Big[\bigwedge_{2 \leqslant a \leqslant p^r} (px_a = 0 \wedge (\exists y_a)(x_a = p^{k-1} y_a)) \\ \wedge \bigwedge_{2 \leqslant a < b \leqslant p^r} (\sim(\exists z)(pz = 0 \wedge (\exists w)(z = p^k w) \wedge x_a = x_b z)) \Big] \end{aligned}$$

says that there are at least $p^r - 1$ elements of order p which are divisible by p^{k-1} and unequal modulo $P_{p,k}$—or, in other words, $\Phi(p, r)$ says that $s_t \geqslant r$.

We define, for each $t \leqslant n$, a set Φ_t of statements. If t is not a prime power then $\Phi_t = \varnothing$; assume then that $t = p^k$.

Case 1. $s_t = 0$. Then Φ_t consists of the single statement

$$\sim(\exists x)(px = 0 \wedge (\exists y)(x = p^{k-1} y)).$$

Case 2. $0 < s_t < \omega$. Then Φ_t consists of the single statement

$$\Phi(p, s_t) \wedge \sim\Phi(p, s_t + 1).$$

Case 3. $s_t = \omega$. Then $\Phi_t = \{\Phi(p, r) \mid r > 0\}$.

Now let $T_M = \bigcup_{2 \leqslant t \leqslant n} \Phi_t \cup \{(\forall x)(nx = 0)\} \cup \{AG\}$ where AG is the standard set of axioms for an Abelian group. It is clear that $M \models T_M$ and that, using Ulm's theorem, if N is a coutable model of T_M then $N \simeq M$. Hence M is $\aleph_0$-categorical. ∎

3. Direct Sums of Finite Groups

The situation with non-Abelian groups is quite different. One might suppose that since, for example, a direct sum of ω copies of Z_6 is $\aleph_0$-categorical, so would be a direct sum of ω copies of S_3 (the symmetric group on three letters.) More generally one might suppose that if one took a direct sum of groups, each selected from the collection of groups of order $\leqslant$ some fixed n, then the group obtained would be $\aleph_0$-categorical. These conclusions would be far from correct.

Theorem 3. *Let $G = \sum_{j\in N}^{\oplus} H_j$ where each H_j is isomorphic to a group G_i of order $\leqslant n$. Then G is $\aleph_0$-categorical iff every G_i which occurs infinitely often is Abelian.*

Proof. Suppose that G_i occurs infinitely often and is not Abelian. Let $a, b \in G_i$ be such that $b^{-1}ab \neq a$; let m be the number of conjugates of a in G_i. Assume that the groups $H_{i_1}, H_{i_2}, \ldots$ are all isomorphic to G_i and that the images of a under these isomorphisms are $a_{i_1}, a_{i_2}, \ldots$. Let $\nu \in G$; then ν can be written as hb_{i_1} where $b_{i_1} \in H_{i_1}$ and $h \in \sum_{i\neq i_1}^{\oplus} H_i$. Hence any conjugate $\nu^{-1}a_{i_1}\nu$ of a_{i_1} can be written $h^{-1}b_{i_1}^{-1}a_{i_1}b_{i_1}h = b_{i_1}^{-1}a_{i_1}b_{i_1}$, so that a has precisely m conjugates in G. More generally, ν can be written as $hb_{i_1}b_{i_2}\cdots b_{i_k}$ where $b_{i_t} \in H_{i_t}$ and $h \in \sum_{i\neq i_1,\ldots,i_k} H_i$ so

$$\nu^{-1}a_{i_1}a_{i_2}\cdots a_{i_k}\nu = (b_{i_1}^{-1}a_{i_1}b_{i_1})(b_{i_2}^{-1}a_{i_2}b_{i_2})\cdots(b_{i_k}^{-1}a_{i_k}b_{i_k})$$

so that $a_{i_1}\cdots a_{i_k}$ has precisely m^k conjugates in $\sum^{\oplus} H_i$.

If we now let $\phi^k(\nu_1)$ be (setting $k' = m^k$)

$$(\exists y_1)(\exists y_2)\cdots(\exists y_{k'})$$
$$\left[\bigwedge_{1\leqslant i<j\leqslant k'} (y_i^{-1}v_1y_i \neq y_j^{-1}v_1y_j) \wedge (\forall z)\left(\bigvee_i z^{-1}v_1z = y_i^{-1}v_1y_i\right)\right]$$

and if we let d_k be $a_{i_1}a_{i_2}\cdots a_{i_k}$ for each k we have an infinite list of distinct first order properties of P^1 and an infinite list of distinct elements of G^1 such that d_i has property ϕ^j in G if and only if $i = j$. Hence by (#) the group G is not $\aleph_0$-categorical.

Conversely if every G_i which occurs infinitely often is Abelian then we can write $G = K_1 \oplus K_2$ where K_1 is an Abelian group of bounded order and K_2 is finite. Since K_1 is $\aleph_0$-categorical (by Theorem 2) and K_2 is $\aleph_0$-categorical (as shown in Section 1) we need only prove that a finite direct sum of $\aleph_0$-categorical groups is $\aleph_0$-categorical. Hence the converse is a consequence of the next theorem. ∎

THEOREM 4. *If $G_1, G_2, \ldots, G_k$ are $\aleph_0$-categorical groups, then*

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$$

is $\aleph_0$-categorical.

Proof. Since each G_i is $\aleph_0$-categorical, we can find sets T_i of statements such that for each i if $M \models T_i$ then $M \simeq G_i$. We wish to say that for each i, the group G contains a subgroup which is a model of T_i. To do this we add to the language of group theory k new unary relation symbols $Q_1, Q_2, \ldots, Q_k$ and in this expanded language say that the elements satisfying Q_i form a model of T_i. To say this we "relativize" the statements of T_i to Q_i. That is to say, we replace, in each statement of T_i, each quantifier of form $(\forall x)(\cdots)$ by $(\forall x)(Q_i(x) \Rightarrow \cdots)$ and each quantifier of form $(\exists x)(\cdots)$ by $(\exists x)(Q_i(x) \wedge \cdots)$; for example, the relativization of the statement $(\forall x)(\exists y)(y^{-1}xy \neq x)$ to Q_i is $(\forall x)(Q_i(x)) \Rightarrow (\exists y)(Q_i(y) \wedge y^{-1}xy \neq x))$.

We define the set T^* of statements to consist of the axioms of group theory together with, for each i, the axioms of T_i relativized to Q_i and the statements

$$\bigwedge_{i \neq j} [(\forall x)(\forall y)(Q_i(x) \wedge Q_j(y) \Rightarrow xy = yx)],$$

$$(\forall x_1)(\forall x_2) \cdots (\forall x_k)(\forall y_1)(\forall y_2) \cdots (\forall y_k)$$
$$\Big[\bigwedge_i (Q_i(x_i) \wedge Q_i(y_i)) \wedge x_1x_2 \cdots x_k = y_1y_2 \cdots y_k \Rightarrow \bigwedge_i (x_i = y_i)\Big],$$

and

$$(\forall x)(\exists y_1)(\exists y_2) \cdots (\exists y_k) \Big(\bigwedge_i Q_i(y_i) \wedge x = y_1y_2 \cdots y_k\Big).$$

A model of T^* consists of a group H together with k distinguished subsets $H_1, H_2, \ldots, H_k$. Since T^* contains the relativizations of the statements of T_i to Q_i it follows that each H_i is a subgroup of H which is a model of T_i and hence is isomorphic to G_i. The additional statements of T^* guarantee that the group H is the direct sum $H_1 \oplus H_2 \oplus \cdots \oplus H_k$, and therefore that $H \simeq G$.

Thus we have shown that the group G, when considered in conjunction with the distinguished subgroups $G_1, G_2, \ldots, G_k$, can be characterized by its first order properties *expressed in an expanded language*. That is to say, if H is any group which, together with k distinguished subsets $H_1, H_2, \ldots, H_k$, has the same first order properties as G, together with $G_1, G_2, \ldots, G_k$; then $H \simeq G$ by an isomorphism which maps each H_i isomorphically onto G_i. We can thus say that the "expanded" group $\langle G, G_1, G_2, \ldots, G_k\rangle$ is $\aleph_0$-categorical. That this implies that G is $\aleph_0$-categorical is a consequence of the lemma below. ▮

LEMMA 1. *Let G be a group; let $H_1, H_2, \dots, H_k$ be subsets of G and let $g_1, g_2, \dots, g_l$ be elements of G. Suppose we add to the language of group theory k new unary relation symbols $Q_1, Q_2, \dots, Q_k$ and l new individual constant symbols $a_1, a_2, \dots, a_l$. Let T^* be the set of statements in this expanded language which are properties of the group G together with the designated subsets $H_1, H_2, \dots, H_k$ and elements $g_1, g_2, \dots, g_l$. Suppose further that if the group G' together with the designated subsets $H_1', H_2', \dots, H_k'$ and elements $g_1', g_2', \dots, g_l'$ also is a model of T^* then there is an isomorphism $f: G \to G'$ such that $f(H_i) = H_i'$ for $1 \leqslant i \leqslant k$ and such that $f(g_i) = g_i'$ for $1 \leqslant i \leqslant l$. Then G is $\aleph_0$-categorical.*

Proof. We define an equivalence relation on G^n by stipulating that if a, $b \in G^n$ then a is logically* equivalent to b if they have precisely the same first order properties (with respect to the expanded language, where each Q_i is interpreted as the set H_i and each a_i is interpreted as the element g_i) as n-tuples of elements of G.

Then, by the general version of the theorem quoted in the introduction, since the hypothesis of the lemma asserts that, as an interpretation of the expanded language, G together with $H_1, H_2, \dots, H_k$ and $g_1, g_2, \dots, g_l$ is $\aleph_0$-categorical, it follows that for each n, the equivalence relation partitions G^n into a finite number of pieces. But if a is logically* equivalent to b, then certainly a is logically equivalent to b. Hence the equivalence relation "a is logically equivalent to b" partitions G^n into fewer pieces than the equivalence relation "a is logically* equivalent to b." Hence $G^n/\equiv_{G,n}$ is finite for every n, so again using the theorem of Engeler, Ryll–Nardzewski, and Svenonius, G is $\aleph_0$-categorical. ∎

It should be noted that we have not actually presented a list of axioms (in the language of group theory) which characterizes the group G of Theorem 4. One could however obtain such a list of axioms, recursively, by taking the set of all statements in the language of group theory which are logical consequences of T^*. In specific cases it is possible to give a nice presentation of the axioms for G in terms of the axioms for the direct summands. The proof of Theorem 2, for example, can be recast in terms of the above and there the axioms for $M = \sum_{2 \leqslant t < n}^{\oplus} M_t$ can be obtained directly from the axioms for the M_t.

4. GROUPS WITH LARGE ABELIAN SUBGROUPS

The results of the previous section suggest that if a group G has a normal Abelian subgroup H of finite index then G is $\aleph_0$-categorical if H is. In this section we examine this conjecture.

We shall start by presenting two examples of such groups.

EXAMPLE 1. Let G be the group generated by $\{x\} \cup \{a_i \mid i \in N\}$ subject to the relations

$$a_i^3 = 1 \qquad a_i a_j = a_j a_i$$
$$x^2 = 1 \qquad x a_i x = a_i^2.$$

We first observe that the subgroup H of G generated by $\{a_i \mid i \in N\}$ is a normal Abelian subgroup of index 2 which is $\aleph_0$-categorical since it is an Abelian group of bounded order. Thus we might conjecture that G is $\aleph_0$-categorical. On the other hand for each i the subgroup of G generated by x and a_i is isomorphic to S_3 so that G is the direct sum of infinitely many copies of S_3 with an amalgamated subgroup, and theorem 3 shows that a direct sum of infinitely many copies of S_3 is not $\aleph_0$-categorical.

Let us show that G is $\aleph_0$-categorical. First note that $G = H \cup Hx$, that $y \in H$ implies $y^3 = 1$, and that $y \in Hx$ implies that $y = a_{i_1} \cdots a_{i_k} x$ so that

$$y^2 = a_{i_1} \cdots a_{i_k} x a_{i_1} \cdots a_{i_k} x = a_{i_1} \cdots a_{i_k} (x a_{i_1} x)(x a_{i_2} x) \cdots (x a_{i_k} x) = 1.$$

Thus $y \in H$ if and only if $y^3 = 1$.

Now let T consist of

(i) the axioms of group theory,

(ii) $(\forall x)(\forall y)[(x^3 = 1 \wedge y^3 = 1) \Rightarrow xy = yx]$,

(iii) $(\exists x_1)(\exists x_2) \cdots (\exists x_k) \left(\bigwedge_i x_i^3 = 1 \wedge \bigwedge_{i \neq j} x_i \neq x_j \right)$, for each k,

(iv) $(\exists x)(\forall y)[(y^3 = 1 \vee (\exists z)(y = zx \wedge z^3 = 1)) \wedge (x^2 = 1) \wedge$
$(\forall y)(y^3 = 1 \Rightarrow xyx = y^2)]$.

It is clear that $G \models T$ and that if $G' \models T$ then $G' \simeq G$. Hence G is $\aleph_0$-categorical. ▮

EXAMPLE 2. Let G be the group generated by $\{x\} \cup \{a_i \mid i \in N\}$ subject to the relations

$$a_i^4 = 1 \qquad a_i a_j = a_j a_i$$
$$x^2 = 1 \qquad x a_i x = a_i a_{i+1}^2$$

We claim that the subgroup H generated by $\{a_i \mid i \in N\}$ is a normal Abelian subgroup of G of finite index, that H is $\aleph_0$-categorical, but that G is not. That H is a normal Abelian subgroup of G of finite index is clear; and, since H is a homomorphic image of a direct sum of infinitely many cyclic groups of order four, H is of bounded order, so that H is $\aleph_0$-categorical.

To show that G is not $\aleph_0$-categorical we need to know that H is actually $\sum_i^{\oplus} \{a_i\}$. (Note that the addition of the element x, and the relations involving it, to $\sum_i^{\oplus} \{a_i\}$ could conceivably disturb this sum. For example, if we were to replace $xa_ix = a_ia_{i+1}^2$ by $xa_ix = a_ia_{i+1}$ we would then have

$$a_i = x(xa_ix)x = (xa_ix)(xa_{i+1}x) = a_ia_{i+1}^2a_{i+2}$$

so that $a_{i+1}^2 = a_{i+2}$, a relation which certainly does not hold in $\sum_i^{\oplus} \{a_i\}$.)

LEMMA 2a. $H = \sum_i^{\oplus} \{a_i\}$.

Proof. It suffices to show that for each word

$$w = a_{i_1}^{c_1}a_{i_2}^{c_2} \cdots a_{i_k}^{c_k}$$

where $i_1 < i_2 < \cdots < i_k$ and where each $c_t \in \{1, 2, 3\}$, there is a homomorphism $h: G \to G^*$ whose kernel does not include w. If some c_t is odd then we define $h: G \to Z_2$ by $h(a_{i_t}) = 1$, $h(a_j) = 0$ for $j \neq i_t$, $h(x) = 0$; it is easy to verify that h is a homomorphism and that w is not in the kernel of h. The case where each $c_t = 2$ is a little more difficult and requires the introduction of a new group G^*.

Consider the group G^* with the presentation $\{a, b, y; a^4 = 1, b^2 = 1, y^2 = 1, ba = ab, ya = ay, yby = ba^2\}$. Given any word in the generators, it it can be written in the form $a^ib^jy^k$ where $0 \leqslant i < 4$, $0 \leqslant j < 2$, and $0 \leqslant k < 2$, so that the number of elements in G^* is at most 16. We wish to show that G^* actually has 16 elements so that no two distinct words of the above form are equal. So we map G^* into the symmetric group S_8 by mapping

$$\begin{aligned} a &\to (1234)(5678) = a^* \\ b &\to (13)(24) = b^* \\ y &\to (15)(26)(37)(48) = y^* \end{aligned}$$

and verifying that a^*, b^*, and y^* satisfy the presenting relations for a, b, and y, and furthermore that there are 16 distinct permutations generated by a^*, b^*, and y^*. From this it follows that G^* has 16 elements. In particular a^2 is not the identity in G^*.

Now define a map $h: G \to G^*$ by $h(a_{i_1}) = a$, $h(a_{i_1-1}) = b$, $h(a_i) = 1$ for $i \neq i_1$, $i_1 - 1$, and $h(x) = y$. This map is a homomorphism which maps $w = a_{i_1}^{c_1}a_{i_2}^{c_2} \cdots a_{i_k}^{c_k}$ to a^2, which, by the paragraph above, is not the identity in G^*. We have thus completed the proof that $H = \sum_i^{\oplus} \{a_i\}$. In particular we know that $a_i^2 = a_j^2$ if and only if $i = j$. ▮

We now proceed to show that G is not $\aleph_0$-categorical. We let $\phi^k(v_1, v_2)$ be

$$(\exists s_1)(\exists s_2) \cdots (\exists s_k)(\exists y)[yv_1y = v_1s_1^2 \wedge ys_1y = s_1s_2^2 \wedge \cdots \wedge ys_{k-1}y = s_{k-1}s_k^2 \wedge s_k = v_2 \wedge y^2 = 1 \wedge s_1^2 \neq 1]$$

and we let d_k be $\langle a_1, a_{1+k}\rangle$ for each k. It suffices to show that d_i has property ϕ^j in G if and only if $i = j$, for we can then apply (#) to conclude that G is not $\aleph_0$-categorical. We first prove the following lemma.

LEMMA 2b. *If $y, r_1, r_2, \ldots, r_k$ are elements of G such that $ya_1y = a_1r_1^2$ $yr_1y = r_1r_2^2, \ldots, yr_{k-1}y = r_{k-1}r_k^2$ and if $y^2 = 1$ and $r_1^2 \neq 1$ then $r_k^2 = a_{1+k}^2$.*

Proof. We first claim that $xa_1x = a_1r_1^2$, $xr_1x = r_1r_2^2, \ldots, xr_{k-1}x = r_{k-1}r_k^2$. For if $y \in H$ then since $y^2 = 1$ we have $ya_1y = a_1$ so that $r_1^2 = 1$ contrary to assumption. On the other hand if $y \in Hx$ then $y = a_1^{i_1}a_2^{i_2} \cdots a_j^{i_j}x$ and a simple calculation of y^2 shows that each i_t is even. But each a_t^2 is in the center of G, hence $y = zx$ where z is in the center of G and has order 2. Hence $yry = (zx)r(zx) = xrx$ for each $r \in G$. Thus the claim is proven.

We now proceed by induction on k. If $k = 1$ then $a_1a_2^2 = xa_1x = a_1r_1^2$ so that $r_1^2 = a_2^2$. Assume that $r_j^2 = a_{1+j}^2$. We first observe, by a direct calculation, that if $r_j \in Hx$ so that $r_j = a_1^{i_1}a_2^{i_2} \cdots a_b^{i_b}x$ then r_j^2 cannot be a_{1+j}^2, and hence that $r_j \in H$. But then either

$$r_j = a_{1+j}a_{i_1}^2a_{i_2}^2 \cdots a_{i_t}^2 \qquad \text{or} \qquad r_j = a_{1+j}^3a_{i_1}^2a_{i_2}^2 \cdots a_{i_t}^2.$$

In the first case,

$$xr_jx = (xa_{1+j}x)(xa_{i_1}x)^2 \cdots (xa_{i_t}x)^2 = a_{1+j}a_{2+j}^2a_{i_1}^2 \cdots a_{i_t}^2 = r_ja_{2+j}^2,$$

and in the second case,

$$xr_jx = (xa_{1+j}x)^3(xa_{i_1}x)^2 \cdots (xa_{i_t}x)^2 = r_ja_{2+j}^2.$$

But $xr_jx = r_jr_{j+1}^2$. Therefore, $r_{j+1}^2 = a_{2+j}^2$, and that proves the lemma. ∎

Now it is clear that d_k has property ϕ^k. Conversely if $\langle a_1, a_{1+i}\rangle$ has property ϕ^j, then by Lemma 2b, $a_{1+i}^2 = a_{1+j}^2$. But Lemma 2a implies that $i = j$. Hence d_i has property ϕ^j if and only if $i = j$. Hence G is not $\aleph_0$-categorical. ∎

These two examples show, on the one hand, that the conjecture at the beginning of this section is not correct, and, on the other hand, that the determination of whether a given group satisfying the hypotheses of the conjecture is $\aleph_0$-categorical may be difficult indeed. It appears to be at least as difficult to establish necessary and sufficient conditions for such groups to be $\aleph_0$-categorical. In the remainder of this section, we present a number of theorems in this direction.

For the sake of avoiding endless repetition of hypotheses, let us call a group G an $n - k$ *group* if G has a normal Abelian subgroup H of exponent n and index k, and if G/H is cyclic. Thus, in particular, an $n-k$ group is metabelian. (We will not discuss the case where G/H is not cyclic.) We are interested in determining for which pairs $\langle n, k\rangle$ it is true that an $n-k$ group must be $\aleph_0$-categorical.

Example 2 shows that a $4-2$ group need not be $\aleph_0$-categorical. Our first result in the other direction was that every $p-2$ group is $\aleph_0$-categorical, for every prime p. We prove here a generalization of this theorem. As with subsequent theorems, the proof is divided into two parts. The first part, separated off as a proposition, provides a group-theoretic analysis of the groups discussed in the theorem; the second part consists of translating the results of this analysis into statements of the first-order predicate calculus.

PROPOSITION 1. *Let G be a $p-k$ group where p is a prime and $k \mid p-1$ so that Z_p contains a primitive k-th root of unity. Let $x \in G$ be such that $G = H \cup Hx \cup Hx^2 \cup \cdots \cup Hx^{k-1}$, and let $1 = \mu_0, \mu_1, \mu_2, \ldots, \mu_{k-1}$ be the k-th roots of unity in Z_p. Then there are subgroups $H_0, H_1, \ldots, H_{k-1}$ each normal in G such that $H = \sum_{0\leqslant t<k}^{\oplus} H_t$ and such that for each $z \in H_t$, $x^{-1}zx = z^{\mu_t}$.*

Proof. For each i, $0 \leqslant t < k$, let $H_t = \{z \in H \mid x^{-1}zx = z^{\mu_t}\}$. Note that $H_t < G$. Furthermore $H_t \lhd G$ for if $z \in H_t$ then, by induction on j, $x^{-j}zx^j = z^{\mu^j}$ (where $\mu = \mu_t$) so that every conjugate of z is a power of z and hence is in H_t. (Note also that if $x^{-1}zx = z^c$ then $z = x^{-k}zx^k = z^{c^k}$ so that $c^k \equiv 1 \pmod p$) so that c must be a k-th root of unity in Z_p.)

We wish to show that $H = \sum_{0\leqslant t<k}^{\oplus} H_t$. Our first observation is that $\sum_{0\leqslant t<k} H_t = \sum_{0\leqslant t<k}^{\oplus} H_t$. To prove this we need only verify that if $z_t \in H_t$ for each t then $z_1 z_2 \cdots z_t = 1$ implies $z_1 = z_2 = \cdots = z_t = 1$. So choose a representation of 1 in which as few as possible $\geqslant 2$ of the z_t are unequal to 1. Suppose that $z_1 = z_2 = \cdots = z_{t_0-1} = 1$ and that $z_{t_0} \neq 1$. Hence

$$z_{t_0}^{\mu_{t_0}} z_{t_0+1}^{\mu_{t_0}} \cdots z_{k-1}^{\mu_{t_0}} = 1.$$

But

$$1 = x^{-1} z_{t_0} z_{t_0+1} \cdots z_{k-1} x = z_{t_0}^{\mu_{t_0}} z_{t_0+1}^{\mu_{t_0+1}} \cdots z_{k-1}^{\mu_{k-1}}.$$

Therefore,

$$1 = z_{t_0+1}^{\mu_{t_0+1}-\mu_{t_0}} \cdots z_{k-1}^{\mu_{k-1}-\mu_{t_0}}$$

is a representation of 1 in which fewer z_t's are unequal to 1. This is, of course, a contradiction.

It remains to show that $H \subseteq \sum_{0 \leqslant t < k} H_t$. Let $z_0 \in H$ and define $z_i = x^{-i} z_0 x^i$ for each $i < k$ so that $z_{i+1} = x^{-1} z_i x$. Then for each t,

$$x^{-1} z_0 z_1^{\mu_t^{-1}} z_2^{\mu_t^{-2}} \cdots z_{k-1}^{\mu_t^{-(k-1)}} x$$
$$= z_1 z_2^{\mu_t^{-1}} z_3^{\mu_t^{-2}} \cdots z_{k-1}^{\mu_t^{-(k-2)}} z_0^{\mu_t^{-(k-1)}} = [z_0 z_1^{\mu_t^{-1}} z_2^{\mu_t^{-2}} \cdots z_{k-1}^{\mu_t^{-(k-1)}}]^{\mu_t}$$

so that

$$z_0 z_1^{\mu_t^{-1}} z_2^{\mu_t^{-2}} \cdots z_{k-1}^{\mu_t^{-(k-1)}} \in H_t \qquad \text{for each } t.$$

Thus for each t,

$$c_t = z_0 z_1^{\mu_t} z_2^{\mu_t^2} \cdots z_{k-1}^{\mu_t^{k-1}} \in \sum_{0 \leqslant t < k} H_t .$$

To show that $z_0 \in \sum_{1 \leqslant t < k} H_t$ it suffices to solve $z_0 = c_0^{b_0} c_1^{b_1} \cdots c_{k-1}^{b_{k-1}}$ for $b_0, b_1, \ldots, b_k$. In other words, to solve the simultaneous linear equations:

$$\left.\begin{array}{llll} b_0 & + b_1 & + \cdots + b_{k-1} & \equiv 1 \\ b_0\mu_0 & + b_1\mu_1 & + \cdots + b_{k-1}\mu_{k-1} & \equiv 0 \\ b_0\mu_0^2 & + b_1\mu_1^2 & + \cdots + b_{k-1}\mu_{k-1}^2 & \equiv 0 \\ & \vdots & & \\ b_0\mu_0^{k-1} & + b_1\mu_1^{k-1} & + \cdots + b_{k-1}\mu_{k-1}^{k-1} & \equiv 0 \end{array}\right\} \pmod p;$$

or in other words to show that the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots 1 \\ \mu_0 & \mu_1 & \mu_2 & \cdots \mu_{k-1} \\ \mu_0^2 & \mu_1^2 & \mu_2^2 & \cdots \mu_{k-1}^2 \\ \vdots & & & \\ \mu_0^{k-1} & \mu_1^{k-1} & \mu_2^{k-1} & \cdots \mu_{k-1}^{k-1} \end{pmatrix}$$

is invertible in Z_p. But this matrix is a Vandermonde matrix whose determinant is $\prod_{0 \leqslant i < j < k} (\mu_j - \mu_i)$ which is not zero.

Hence $H \subseteq \sum_{0 \leqslant t < k} H_t$ and therefore $H = \sum^{\oplus}_{0 \leqslant t < k} H_t$ as claimed. ∎

Thus to each $p - k$ group G where p is a prime and $k \mid p - 1$ we can assign a sequence $\langle m_0, m_1, \ldots, m_{k-1} \rangle$ of cardinals $\leqslant \aleph_0$ by setting m_t equal to the number of basis elements in H_t.

We note that for each such sequence there is a corresponding $p - k$ group, namely the group generated by $\{x\} \cup \{a_{t_i} \mid 0 \leqslant t < k \wedge i < m_t\}$ subject to the relations $x^k = 1$, $a_{t_i}^p = 1$, $aa' = a'a$, $x^{-1} a_{t_i} x = a_{t_i}^{\mu_t}$. We also note that the group G is completely determined by this sequence. We now must show that these invariants can be carried into the first-order predicate calculus.

THEOREM 5. *Let G be a $p - k$ group where p is a prime and $k \mid p - 1$. Then G is $\aleph_0$-categorical.*

Proof. We will present a set of axioms for the group G formulated in the language of group theory augmented by k unary relation symbols $R_0, R_1, \ldots, R_{k-1}$ and a constant symbol a. These axioms will uniquely determine G (together with $H_0, H_1, \ldots, H_{k-1}$ and x) so that applying Lemma 1 we conclude that G is $\aleph_0$-categorical.

The fact that G satisfies the axioms given below is a consequence of Proposition 1.

The axioms T_G for G will include the following:

(i) the axioms for group theory,

(ii) $\bigwedge_t (\forall y)(R_t(y) \Rightarrow y^p = 1)$,

(iii) $\bigwedge_{t_1, t_2} (\forall y_1)(\forall y_2)(R_{t_1}(y_1) \wedge R_{t_2}(y_2) \Rightarrow y_1 y_2 = y_2 y_1)$,

(iv) $(\forall y_0)(\forall y_1) \cdots (\forall y_{k-1}) \left[\left(\bigwedge_t R_t(y_t) \wedge y_0 y_1 \cdots y_{k-1} = 1\right) \Rightarrow \bigwedge_t y_t = 1\right]$,

(v) $\bigwedge_t (\forall y)(\forall z)(R_t(y) \wedge R_t(z) \Rightarrow R_t(yz) \wedge R_t(y^{-1}))$,

(vi) $\bigwedge_t \bigwedge_{1 \leqslant i < k} \sim R_t(a^i)$,

(vii) $(\forall x)(\exists y_0)(\exists y_1) \cdots (\exists y_{k-1}) \left[\bigwedge_t R_t(y_t) \wedge \bigvee_{0 \leqslant j < k} x = y_0 y_1 \cdots y_{k-1} a^j\right]$,

(viii) $(\forall x)(\forall y) \left(\left[\bigvee_t R_t(x) \wedge \bigvee_t R_t(y)\right] \Rightarrow \left[\bigwedge_{0 \leqslant i < j < k} xa^i \neq ya^j\right]\right)$,

(ix) $a^k = 1 \wedge (\forall z) \left[\bigwedge_t (R_t(z) \Rightarrow a^{-1}za = z^{\mu_t})\right]$.

The remaining axioms depend on the group G; more specifically on the sequence $\langle m_0, m_1, \ldots, m_{k-1} \rangle$ associated with G. For each t, if m_t is finite we add the following axiom:

(x) $(\exists!^{p^{m_t}} y)\, R_t(y)$.

For each t for which m_t is infinite we add the statements

$(\mathrm{xi})_n$ $(\exists \geqslant^n y)\, R_t(y)$.

It is clear that this set of statements does what was claimed; hence by Lemma 1 the group G is $\aleph_0$-categorical. ∎

In attempting to generalize Theorem 5 there are several requirements that could be relaxed. One, we could not insist that p be a prime. Two, we could not insist that $k \mid p-1$. (Three, we could not insist that G/H be cyclic. As mentioned earlier this case will not be considered here.)

We shall first consider the situation when we relax the requirement that p be a prime.

THEOREM 6. *Let n be square-free and assume that $k \mid p-1$ for each $p \mid n$. Then every $n-k$ group G is $\aleph_0$-categorical.*

Proof. For each prime $p \mid n$, let $H^p = \{h \in H \mid h^p = 1\}$. Let $x \in G$, $x^k = 1$, be such that $G = H \cup Hx \cup \cdots \cup Hx^{k-1}$. Then since $(x^{-1}hx)^p = x^{-1}h^px = 1$ and since $H \lhd G$ it follows that each $H^p \lhd G$. Also $H = \sum_{p\mid n}^{\oplus} H^p$ and the group G_p generated by H^p and x is a $p-k$ group which is $\aleph_0$-categorical by Theorem 5. Accordingly, with each G_p we have associated invariants $\langle m_0{}^p, m_1{}^p, \dots, m_{k-1}^p\rangle$.

We will present a set of axioms for the group G in the language of group theory augmented by rk unary relation symbols $\{R_t{}^i \mid 0 \leqslant t < k \wedge 1 \leqslant i \leqslant r\}$, where $n = p_1 p_2 \cdots p_r$, and one constant symbol a.

For each i, $1 \leqslant i \leqslant r$, we write down, using the relation symbols $R_0{}^i, R_1{}^i, \dots, R_{k-1}^i$ and the constant symbol a, the set of statements $T_{G_{p_i}}$ given for a $p_i - k$ group with invariants $\langle m_0^{p_i}, m_1^{p_i}, \dots, m_{k-1}^{p_i}\rangle$ in the proof of the preceding theorem. We let T be the union of these statements, deleting each axiom (vii), together with the statements

(xii) $$(\forall x)(\exists y_0{}^1)(\exists y_1{}^1) \cdots (\exists y_{k-1}^1)(\exists y_1{}^2)(\exists y_2{}^2) \cdots (\exists y_{k-1}^2) \cdots$$
$$(\exists y_0{}^r)(\exists y_1{}^r) \cdots (\exists y_{k-1}^r) \left[\bigwedge_{t,i} R_t{}^i(y_t{}^i) \wedge \bigvee_{0 \leqslant j < k} \left(x = \left(\prod_{i,t} y_t{}^i\right) \cdot a^j\right)\right]$$

(xiii) $$(\forall y)\Big(\bigwedge_{\langle t,i\rangle \neq \langle s,j\rangle} (R_t{}^i(y) \wedge R_s{}^j(y) \Rightarrow y = 1)\Big)$$

(xiv) $$(\forall x)(\forall y)\Big(\bigwedge_{t,i,s,j} (R_t{}^i(x) \wedge R_s{}^j(y) \Rightarrow xy = yx)\Big)$$

(xv) $$(\forall y_1)(\forall y_2) \cdots (\forall y_r)$$
$$\Big(\bigwedge_i \bigvee_t (R_t{}^i(y_i) \wedge y_1 y_2 \cdots y_r = 1) \Rightarrow (y_1 = 1 \wedge y_2 = 1 \wedge \cdots \wedge y_r = 1)\Big).$$

It is clear that this set of statements is $\aleph_0$-categorical; and hence by Lemma 1 of Section 3 the group G is $\aleph_0$-categorical. ▮

Before we try to further generalize Theorem 6 let us consider the case where $k = 2$. Since $2 \mid p - 1$ for every odd prime number p, it follows from the theorem above that every $n - 2$ group is $\aleph_0$-categorical so long as n is odd and square-free. If we can prove that every $2 - 2$ group is $\aleph_0$-categorical then using the argument in the proof of Theorem 6 we will obtain the following result.

COROLLARY. *Every $n - 2$ group is $\aleph_0$-categorical for n square-free.*

This result is best possible for we shall subsequently prove that if n is not square-free then there is an $n - 2$ group which is not $\aleph_0$-categorical.

PROPOSITION 2. *Let G be a $2 - 2$ group. Then there are subgroups Z_1, Z_2, Z_3 of H such that*

(1) $H = Z_1 \oplus Z_2 \oplus Z_3$

(2) $Z_1 \oplus Z_2 = \{z \in H \mid xzx = z\}$

(3) $Z_2 = \{z \in H \mid (\exists y)(y \in H \wedge xyx = yz)\}$

(4) *The map $f: Z_3 \to Z_2$ defined by $f(y) = z$ iff $xyx = yz$ is an isomorphism.*

Proof. Let $Z = \{z \in H \mid xzx = z\}$ and let

$$Z_2 = \{z \in H \mid (\exists y)(y \in H \wedge xyx = yz)\}.$$

If $z \in Z_2$, and we choose $y \in H$ so that $xyx = yz$ then $y = x(xyx)x = (xyx)(xzx) = yz\,xzx$ so that $xzx = z$ and $z \in Z$. Since Z and Z_2 are both groups it follows that $Z_2 < Z$. We can find a subgroup $Z_1 < Z$ such that $Z = Z_1 \oplus Z_2$ since we are dealing with vector spaces. Note that Z_1 and Z_2 are normal subgroups of G. Now choose Z_3 so that $H = Z \oplus Z_3$. We claim that the map defined in (4) is an isomorphism. The map is clearly a homomorphism; also if $f(y_1) = z$ and $f(y_2) = z$ then $xy_1y_2x = y_1y_2$ so that $y_1y_2 \in Z$. Therefore, $y_1y_2 = 1$ so that $y_1 = y_2$. Finally if $z \in Z_2$ and $x^{-1}yx = yz$ for some y, we write $y = z_1y_1$ where $z_1 \in Z$ and $y_1 \in Z_3$. But then $xy_1x = y_1z$ so that $f(y_1) = z$. Therefore, f is an isomorphism. ∎

Thus with each $2 - 2$ group we can associate the invariants $\langle m_1, m_2\rangle$ by stipulating that m_i is the cardinality of a basis of Z_i. Note that for each pair $\langle m_1, m_2\rangle$ of cardinals $\leqslant \aleph_0$ there is a unique (up to isomorphism) $2 - 2$ group which has those invariants.

THEOREM 7. *Every $2 - 2$ group G is $\aleph_0$-categorical.*

Proof. We will present a set of axioms for the group G formulated in the language of group theory augmented by 3 unary relation symbols R_1, R_2, R_3 and a constant symbol a:

(i) the axioms for group theory,

(ii) $(\forall y)(R_1(y) \vee R_2(y) \vee R_3(y) \Rightarrow y^2 = 1)$,

(iii) $\bigwedge_{t_1, t_2} (\forall x)(\forall y)(R_{t_1}(x) \wedge R_{t_2}(y) \Rightarrow xy = yx)$,

(iv) $(\forall x)(\forall y)(\forall z)((R_1(x) \wedge R_2(y) \wedge R_3(z) \wedge xyz = 1) \Rightarrow (x = 1 \wedge y = 1 \wedge z = 1))$,

(v) $\bigwedge_t (\forall x)(\forall y)(R_t(x) \wedge R_t(y) \Rightarrow R_t(xy))$,

(vi) $\left[\bigwedge_t \sim R_t(a)\right] \wedge a^2 = 1$,

(vii) $(\forall x)(\exists y_1)(\exists y_2)(\exists y_3)$
$[R_1(y_1) \wedge R_2(y_2) \wedge R_3(y_3) \wedge (x = y_1 y_2 y_3 \vee x = y_1 y_2 y_3 a)]$,

(viii) $(\forall x)(\forall y)\left[\left(\bigvee_t R_t(x) \wedge \left(\bigvee_t R_t(y)\right)\right) \Rightarrow x \neq ya\right]$,

(ix) $(\forall y)(R_1(y) \vee R_2(y) \Rightarrow aya = y)$,

(x) $(\forall y)(R_3(y) \Rightarrow (\exists z)(R_2(z) \wedge aya = yz))$,

(xi) $(\forall z)(R_2(z) \Leftrightarrow (\exists y)(R_3(y) \wedge aya = yz))$.

The remaining axioms depend on m_1 and m_2. For each i, if m_i is finite we add the axiom

(xii) $(\exists !^{2^{m_i}} z)\, R_i(z)$

whereas if m_i is infinite we add the axioms

$(\text{xiii})_n$ $(\exists \geqslant^n z)\, R_i(z)$.

It is clear from Proposition 2 that this set of statements is $\aleph_0$-categorical; hence using Lemma 1 of Section 3, G is $\aleph_0$-categorical. ▌

THEOREM 8a. *For each odd prime p and each $k > 1$ there is a $p^k - 2$ group G which is not $\aleph_0$-categorical.*

Proof. We first present the group G; we then show that it is what it appears to be, and we finally show that it is, in fact, not $\aleph_0$-categorical.

The group G has as its generators $\{x\} \cup \{a_{i+1} \mid i \in N\}$ subject to the relations $a_i^{p^k} = 1$, $x^2 = 1$, $a_i a_j = a_j a_i$, $x a_{2i+1} x = a_{2i+1} a_{2i+2}^{p^{k-1}}$, $x a_{2i+2} x = a_{2i+2}^{-1} a_{2i+3}^{p}$.

To verify that the group is indeed what it purports to be we take a word $g = a_1^{\epsilon_1} a_2^{\epsilon_2} \cdots a_r^{\epsilon_r} x^{\epsilon} = 1$, where each $\epsilon_j < p^k$ and $\epsilon < 2$, and show that ϵ and each ϵ_j is 0. We first map G homomorphically to the group $\{\bar{x}; \bar{x}^2 = 1\}$ by sending each a_i to 1, so that g is mapped to $\bar{x}^\epsilon$. Thus since $g = 1$ we can conclude that $\epsilon = 0$.

Let t be such that $\epsilon_t \neq 0$ but $\epsilon_s = 0$ for $s < t$. Define groups G_0 and G_1 by

$$G_0 = \{a, b, c;\ a^{p^k} = b^p = c^2 = 1, ab = ba, cac = a^{-1}, cbc = ba^{p^{k-1}}\}$$

and

$$G_1 = \{a, b, c;\ a^{p^k} = b^{p^{k-1}} = c^2 = 1, ab = ba, cac = a, cbc = b^{-1}a^p\}.$$

If t is even, we define $f\colon G \to G_0$ by $f(x) = c, f(a_t) = a, f(a_{t-1}) = b$, and $f(a_j) = 1$ for $j \neq t$ or $t - 1$. This map defines a homomorphism such that $f(g) = a^{\epsilon_t}$. Thus it suffices to show that if $a^\epsilon = 1$ in G_0, where $\epsilon < p^k$, then $\epsilon = 0$.

If t is odd, we define $f\colon G \to G_1$ by $f(x) = c, f(a_t) = a, f(a_{t-1}) = b$, and $f(a_j) = 1$ for $j \neq t$ or $t - 1$. (If $t = 1$, G is mapped to $\{a, c\}$.) This map defines a homomorphism such that $f(g) = a^{\epsilon_t}$. Thus it suffices to show that if $a^\epsilon = 1$ in G_1, where $\epsilon < p^k$, then $\epsilon = 0$.

To show that if $a^\epsilon = 1$ in G_0 then $\epsilon = 0$, we introduce the group

$$G_2 = \{\bar{a}, \bar{c};\ \bar{a}^{p^k} = 1, \bar{c}^2 = 1, \overline{cac} = \bar{a}^{-1}\}$$

in which $\bar{a}^\epsilon \neq 0$ for $\epsilon \neq 0$ since G_2 is isomorphic to the subgroup of S_{p^k} generated by

$$(1\ 2\ 3\ \cdots\ p^k) \qquad \text{and} \qquad \begin{pmatrix} 1 & 2 & \cdots & p^k \\ p^k & p^k - 1 & \cdots & 1 \end{pmatrix}.$$

We define a map $f_0\colon G_0 \to G_2$ by $f_0(a) = \bar{a}$, $f_0(c) = \bar{c}$, $f_0(b) = \bar{a}^s$ where $s = ((p - 1)/2)(p^{k-1})$. This map defines a homomorphism such that $f(a^\epsilon) = \bar{a}^\epsilon$.

Similarly, to show that if $a^\epsilon = 1$ in G_1 then $\epsilon = 0$, we define a map

$$f_1\colon G_1 \to G_3 = \{\bar{a}, \bar{c};\ \bar{a}^{p^k} = 1, \bar{c}^2 = 1, \overline{cac} = \bar{a}\}$$

by $f_1(a) = \bar{a}$, $f_1(c) = \bar{c}$, $f_1(b) = \bar{a}^s$ where $s = p^k + p/2$. This map defines a homomorphism such that $f(a^\epsilon) = \bar{a}^\epsilon$.

These representations of G_0 and G_1 conclude the proof that if

$$a_1^{\epsilon_1} a_2^{\epsilon_2} \cdots a_r^{\epsilon_r} x^\epsilon = 1$$

in G then $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_r = \epsilon = 0$, and thus the group G is what it appears to be.

We now show that the group G is not $\aleph_0$-categorical. Define $\phi^{2n}(\nu_1, \nu_2)$ to be

$$(\exists s_1)(\exists s_2) \cdots (\exists s_{2n})(\exists y)[y\nu_1 y = \nu_1 s_1^{p^{k-1}} \wedge ys_1y = s_1^{-1}s_2{}^p \wedge ys_2y = s_2 s_3^{p^{k-1}}$$
$$\wedge \cdots \wedge ys_{2n-1}y = s_{2n-1}^{-1}s_{2n}^p \wedge ys_{2n}y = s_{2n}\nu_2^{p^{k-1}} \wedge y^2 = 1 \wedge s_1^{p^{k-1}} \neq 1]$$

and define $\phi^{2n+1}(\nu_1, \nu_2)$ in an analogous way. It is clear that in these groups $d_j = \langle a_1, a_{j+2}\rangle$ has property ϕ^j. To show that the converse is true, we prove by induction, as in Example 2, that if $\langle a_1, h\rangle$ has property ϕ^j then if j is even, $h^p = a_{j+2}^p$, and if j is odd, $h^{p^2} = a_{j+2}^{p^2}$. This combined with the information above guarantees that d_i has property ϕ^j if and only if $i = j$. Hence the group G is not $\aleph_0$-categorical.

THEOREM 8b. *For each $k > 1$ there is a $2^k - 2$ group G which is not $\aleph_0$-categorical.*

Proof. The group G is defined analogously to the group of Example 2. Its generators are $\{x\} \cup \{a_i \mid i \in N\}$ subject to the relations $x^2 = 1$, $a_i^{2^k} = 1$, $a_i a_j = a_j a_i$, $xa_ix = a_i a_{i+1}^{2^{k-1}}$. We first show that if each $\epsilon_j < p^k$ and $\epsilon < 2$ and $a_1^{\epsilon_1}a_2^{\epsilon_2} \cdots a_k^{\epsilon_k}x^{\epsilon} = 1$ then $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_k = \epsilon = 0$. By mapping G to Z_2 we conclude that $\epsilon = 0$. Now map the group G to the group

$$G^* = \{a, b, y; a^{2^k} = 1, b^2 = 1, y^2 = 1, ba = ab, ya = ay, yby = ba^{2^{k-1}}\}$$

by sending $a_t \to a$, $a_{t-1} \to b$, $a_i \to 1$ for $i \neq t, t-1$, $x \to y$ where $\epsilon_t \neq 0$ but $\epsilon_s = 0$ for $s < t$. This defines a homomorphism in which the given element is mapped to a^{ϵ_t}. So we need only verify that, in the group G^*, $a^\epsilon \neq 1$ unless $\epsilon = 0$. This can be shown as in Example 2 by mapping G^* to the symmetric group $S_{2^{k+1}}$ and showing thereby that it actually has 2^{k+2} elements.

Let $\phi^e(\nu_1, \nu_2)$ be

$$(\exists s_1)(\exists s_2) \cdots (\exists s_e)(\exists y)[y\nu_1 y = \nu_1 s_1^{2^{k-1}} \wedge ys_1y = s_1 s_2^{2^{k-1}} \wedge \cdots \wedge ys_{e-1}y$$
$$= s_{e-1}s_e^{2^{k-1}} \wedge s_e = \nu_2 \wedge y^2 = 1 \wedge s_1^{2^{k-1}} \neq 1]$$

and let d_e be $\langle a_1, a_{1+e}\rangle$ for each e. We then show by induction on e that if $\langle a_1, r_e\rangle$ has property ϕ^e then $r_e^{2^{k-1}} = a_{1+e}^{2^{k-1}}$, and conclude from this that d_i has property ϕ^j if and only if $a_{1+i}^{2^{k-1}} = a_{1+j}^{2^{k-1}}$, which, by our analysis above, can happen only if $i = j$. Hence the group G is not $\aleph_0$-categorical. ∎

We are now able to summarize our results for $n - 2$ groups in the following theorem.

THEOREM 9. *For each $n \in N$, every $n - 2$ group is $\aleph_0$-categorical if and only if n is square-free.*

Proof. That every $n - 2$ group is $\aleph_0$-categorical for n square-free was pointed out earlier as a corollary to Theorem 6. If n is not square-free, say $p^a \mid n$ we can use the result of Theorem 8 to construct a $p^a - 2$ group which is not $\aleph_0$-categorical and, by adding in additional generators which commute with x, expand it to an $n - 2$ group which is not $\aleph_0$-categorical. ▮

This completes our discussion of $n - 2$ groups. The question that remains unanswered about such groups can be stated as follows: Given an n which is not square-free and an $n - 2$ group G what are necessary and sufficient conditions that the group must satisfy in order for it to be $\aleph_0$-categorical?

We turn our attention now to the case of $n - k$ groups where $k \nmid p - 1$; we shall first consider the case where $k = 3$, leaving the general case for later. For those primes p for which $3 \mid p - 1$ any $p - 3$ group is $\aleph_0$-categorical by Theorem 5.

We begin by a group-theoretic analysis of $p - 3$ groups where $3 \nmid p - 1$.

PROPOSITION 3. *Let G be a $p - 3$ group where p is a prime $\neq 3$, and $3 \nmid p - 1$. Then there are subgroups Z_1 and B of H such that*

(i) $Z_1 = \{z \in H \mid (\exists a)(\exists b)(a \in H \wedge b \in H \wedge x^{-1}ax = ab \wedge x^{-1}bx = a^{-3}b^{-2}z)\} = \{z \in H \mid x^{-1}zx = z\}$

(ii) $B = \{b \in H \mid (\exists d)(d \in H \wedge x^{-1}bx = bd \wedge x^{-1}dx = b^{-3}d^{-2})\}$

(iii) $Z_1 \lhd G, B \lhd G$

(iv) $H = Z_1 \oplus B$

(v) $B = \sum_{i \in I}^{\oplus} B_i$ *where, for each $i \in I$,*

$B_i \lhd G$ *and B_i is two-dimensional.*

Proof. Let $Z = \{z \in H \mid x^{-1}zx = z\}$ and define Z_1 as in (i). We first note that both Z and Z_1 are subgroups of G. If $z \in Z_1$ and we choose a and b as in (i), then $x^{-2}ax^2 = (ab)(a^{-3}b^{-2}z) = a^{-2}b^{-1}z$ so that

$$a = x^{-3}ax^3 = (a^{-2}b^{-2})(a^3b^2z^{-1})(x^{-1}zx) = (az^{-1})(x^{-1}zx). \quad \text{Hence } x^{-1}zx = z$$

and $Z_1 < Z$. Also since Z is in the center of G, we know that $Z_1 \lhd G$.

Define B as in (ii). We first observe that $Z + B = Z \oplus B$. Indeed if $b \in B$ then $x^{-1}bx = bd$ and $x^{-1}dx = b^{-3}d^{-2}$; and, on the other hand, if $b \in Z$ then $x^{-1}bx = b$ so that $d = 1$ and $b^{-3} = 1$, which is impossible (since $p \neq 3$.) Second, $H \subseteq Z \oplus B$. For if $a \in H$ we obtain elements b and c from

$x^{-1}ax = ab$ and $x^{-1}bx = bc$. But then $x^{-2}ax^2 = (ab)(bc)$ and $a = x^{-3}ax^3 = (ab)(bc)^2 \cdot x^{-1}cx$ so that $x^{-1}cx = b^{-3}c^{-2}$. Hence $x^{-1}(a^3b^3c)x = a^3b^3b^3c^3b^{-3}c^{-2} = a^3b^3c$ so that $a^3b^3c \in Z$. Also $x^{-1}b^{-3}c^{-1}x = b^{-3}c^{-3}b^3c^2 = (b^{-3}c^{-1})b^3$ and $x^{-1}b^3x = b^3c^3 = (b^{-3}c^{-1})^{-3}(b^3)^{-2}$ so that $b^{-3}c^{-1} \in B$. Therefore

$$(a^3b^3c)(b^{-3}c^{-1}) \in Z \oplus B,$$

i.e., $a^3 \in Z \oplus B$. Therefore, $a \in Z \oplus B$ so that $H = Z \oplus B$. But $a^3b^3c \in Z_1$ since $x^{-1}(ac^{-1})x = abb^3c^2 = (ac^{-1})(b^4c^3)$ and $x^{-1}(b^4c^3)x = (b^4c^4)(b^{-9}c^{-6}) = (ac^{-1})^{-3}(b^4c^3)^{-2}(a^3b^3c)$. Hence $Z_1 = Z$ and $H = Z_1 \oplus B$.

To show that $B \lhd G$ it suffices to show that if $b \in B$ and if $x^{-1}bx = bd$ then $d \in B$. We shall actually show that $b^id^j \in B$ for every i and j. For if we set $b_0 = b^id^j$, then for $d_0 = b^{-3j}d^{i-3j}$ we can verify directly that $x^{-1}b_0x = b_0d_0$ and that $x^{-1}d_0x = b_0^{-3}d_0^{-2}$. Hence $b^id^j \in B$ for each i and j, so that $B \lhd G$.

In the paragraph above we have shown somewhat more than we have claimed. First of all we have shown that if $b \in B$ and $x^{-1}bx = bd$ then $\langle b, d\rangle \lhd G$. Second, if we define the function $f: \langle b, d\rangle \to \langle b, d\rangle$ by mapping b^id^j to $b^{-3j}d^{i-3j}$ then f is a $1 - 1$ function; for if $\langle b, d\rangle$ is really two-dimensional then clearly $f(b^id^j) = f(b^kd^l)$ if and only if $i = k$ and $d = l$, so we need only concern ourselves with the case where $\langle b, d\rangle$ is one dimensional. But if $d = b^r$ then $x^{-1}bx = b^{1+r}$ so that $x^{-1}\,dx = x^{-1}b^rx = b^{r(1+r)}$ and at the same time $x^{-1}\,dx = b^{-3}d^{-2} = b^{-3-2r}$ so that $r(1 + r) \equiv -3 - 2r \pmod p$ or $r^2 + 3r + 3 \equiv 0 \pmod p$ or $(r + 1)^3 \equiv 1 \pmod p$. But since $3 \nmid p - 1$ we conclude that $r + 1 = 1$ so that $r = 0$, but $r = 0$ does not satisfy $r^2 + 3r + 3 \equiv 0 \pmod p$. Therefore, $\langle b, d\rangle$ cannot be one-dimensional.

Thus the function f defined above is $1 - 1$; but since its domain and range are of equal finite size, this function is also onto.

Now let $B = \{b_0, b_1, b_2, \ldots\}$ be an enumeration of B and let d_i be defined by $x^{-1}b_ix = b_id_i$ for each i. We proceed inductively to define B_n for each $n \in N$. Define $B_0 = \langle b_0, d_0\rangle$. Assume that B_n has been defined and that b_{i_n} is the first element of B not in $\sum_{i=0}^n B_i$. Define $B_{n+1} = \langle b_{i_n}, d_{i_n}\rangle$. (If $B = \sum_{i=0}^n B_i$ then $B_{n+1} = 1$.) Clearly $B = \sum_{n\in N} B_n$. We need only show that $B = \sum_{n\in N}^{\oplus} B_n$, and to prove this it suffices to show that $\sum_{i=0}^n B_i = \sum_{i=0}^{n\oplus} B_i$ for each $n \in N$—and this we prove by induction. Assume then that $\sum_{i=0}^n B_i = \sum_{i=0}^{n\oplus} B_i$ and let $c \in \sum_{i=0}^{n\oplus} B_i$. If we write $c = c_0c_1 \cdots c_n$ where $c_i \in B_i$ for each i, then, by the preceding paragraph we can find $a_i \in B_i$ such that $x^{-1}a_ix = a_ic_i$ for each i. But then with $a = a_0a_1 \cdots a_n$ we have $x^{-1}ax = ac$. If in addition $c \in B_{n+1}$ then, again by the preceding paragraph, $x^{-1}a_1x = a_1c$ where $a_1 \in B_{n+1}$. But then $x^{-1}cx = a^{-3}c^{-2}$ and $x^{-1}cx = a_1^{-3}c^{-2}$ so that $a = a_1$ and hence $a_1 \in \sum_{i=0}^{n\oplus} B_i$. But $c \in \sum_{i=0}^{n\oplus} B_i$ and hence $\langle a_1, c\rangle \in \sum_{i=0}^{n\oplus} B_i$. The argument of the preceding peragraphs shows that since $x^{-1}a_1x = a_1c$, $\langle a_1, c\rangle$ cannot be one-dimensional. Hence $\langle a_1, c\rangle = B_{n+1} \subseteq \sum_{i=0}^{n\oplus} B_i$, which is impossible, unless of course $B_{n+1} = 1$, in which case the induction at this step is somewhat simpler. This completes the proof. ▮

It is clear from Proposition 3 that to each $p - 3$ group G, where p is a prime $\neq 3$ such that $3 \nmid p - 1$, we can assign the invariants $\langle e_1, e_2 \rangle$ where e_1, e_2 are the number of basis elements of Z_1 and B, respectively. That is to say, for each ordered pair $\langle e_1, e_2 \rangle$ there is a unique $p - 3$ group G which has subgroups Z_1 and B meeting the requirements of Proposition 3 and whose dimensions are, respectively, e_1 and e_2. That this results in Theorem 10a is clear.

THEOREM 10a. *For every prime $p \neq 3$ for which $3 \nmid p - 1$, every $p - 3$ group G is $\aleph_0$-categorical.*

Proof. As in earlier theorems we need only translate the results of Proposition 3 into statements of the predicate calculus. That this can be done should be clear to the reader and we leave it as an exercise. ▮

PROPOSITION 4. *Let G be a $3 - 3$ group. Then there are subgroups Z_1, Z_2, B_1, B_2, C of H such that*

(i) $H = Z_1 \oplus Z_2 \oplus B_1 \oplus B_2 \oplus C$

(ii) $Z_1 \oplus Z_2 = \{z \in H \mid x^{-1}zx = z\}$

(iii) $Z_2 = \{z \in H \mid (\exists b)(b \in H \wedge x^{-1}bx = bz) \wedge x^{-1}zx = z\}$

(iv) $Z_1 \oplus Z_2 \oplus B_1 \oplus B_2 = \{b \in H \mid (\exists z)(x^{-1}bx = bz \wedge x^{-1}zx = z)\}$

(v) $B_2 \oplus Z_2 = \{b \in H \mid (\exists a)(x^{-1}ax = ab)\}$

(vi) *The map $g\colon C \to B_2$ defined by $g(a) = b$ if $x^{-1}ax = ab$ is an isomorphism.*

(vii) *The map $f\colon B_1 \oplus B_2 \to Z_2$ defined by $f(b) = z$ if $x^{-1}bx = bz$ is an isomorphism.*

Proof. Define $Z = \{z \in H \mid x^{-1}zx = z\}$ and define Z_2 by (iii). It follows that Z_2 and Z are subgroups of H and that $Z_2 < Z$. Define Z_1 so that $Z = Z_1 \oplus Z_2$, using the fact that Z is a vector space.

Let $A = \{b \in H \mid (\exists z)(x^{-1}bx = bz \wedge x^{-1}zx = z)\}$. Clearly $Z < A$. Let $D = \{b \in H \mid (\exists a)(a \in H \wedge x^{-1}ax = ab)\}$. If $b \in D$ and $x^{-1}bx = bz$ then it follows that $x^{-1}zx = z$ (since $x^{-2}ax^2 = (ab)(bz)$ and hence $a = x^{-3}ax^3 = (ab)(bz)^2x^{-1}zx$), so that $b \in A$. Thus $D < A$. Furthermore $D \cap Z = Z_2$. Define B_2 so that $Z_2 \oplus B_2 = D$ and define B_1 so that

$$Z_1 \oplus Z_2 \oplus B_1 \oplus B_2 = A.$$

Define $f: B_1 \oplus B_2 \to Z_2$ by $f(b) = z$ if $x^{-1}bx = bz$. This map is clearly a homomorphism; it is $1-1$ since if $x^{-1}b_1x = b_1z$ and $x^{-1}b_2x = b_2z$ then $x^{-1}b_1b_2^{-1}x = b_1b_2^{-1}$ so that $b_1b_2^{-1} \in (B_1 \oplus B_2) \cap Z$ and therefore $b_1 = b_2$; and it is onto, since if $z \in Z_2$ and $x^{-1}bx = bz$ then $b \in A$ so that $b = z_1b_1$ where $z_1 \in Z_1 \oplus Z_2$ and $b_1 \in B_1 \oplus B_2$. But then

$$x^{-1}b_1x = x^{-1}bz_1^{-1}x = bzz_1^{-1} = b_1z$$

so that $f(b_1) = z$. Hence f is an isomorphism between $B_1 \oplus B_2$ and Z.

Let $\{b_i \mid i \in I\}$ be a basis for B_2, and for each $i \in I$ let a_i be such that $x^{-1}a_ix = a_ib_i$ and $a_i \in H$. Let C be the subgroup of H generated by $\{a_i \mid i \in I\}$. It is then clear that (vi) holds. We now show that $(Z \oplus B_1 \oplus B_2) + C = Z \oplus B_1 \oplus B_2 \oplus C$. Indeed if $a \in C$ then $a = a_{i_1}^{r_1}a_{i_2}^{r_2} \cdots a_{i_k}^{r_k}$ so that

$$x^{-1}ax = (b_{i_1}^{r_1}b_{i_2}^{r_2} \cdots b_{i_k}^{r_k})a;$$

i.e., $(x^{-1}ax)a^{-1} \in B_1 \oplus B_2$. On the other hand if $a \in Z \oplus B_1 \oplus B_2$, say $a = zb_1$ where $z \in Z$ and $b_1 \in B_1 \oplus B_2$, then $x^{-1}ax = z(b_1z_1) = az_1$ where $z_1 \in Z$; i.e. $(x^{-1}ax)a^{-1} \in Z$. But $Z \cap (B_1 \oplus B_2) = 1$. Therefore, $x^{-1}ax = a$ so that $a \in Z$. Therefore, $b_{i_1}^{r_1}b_{i_2}^{r_2} \cdots b_{i_k}^{r_k} = 1$ which is impossible unless $r_1 = r_2 = \cdots = r_k = 0$ and hence $a = 1$. Therefore, $(Z \oplus B_1 \oplus B_2) + C = Z \oplus B_1 \oplus B_2 \oplus C$.

Finally $H = Z_1 \oplus Z_2 \oplus B_1 \oplus B_2 \oplus C$. For if $a \in H$ we write $x^{-1}ax = ab$ and $x^{-1}bx = bz$ where $x^{-1}zx = z$. Since $b \in D = Z_2 \oplus B_2$ we can write $b = b_2z_2$ where $b_2 \in B_2$ and $z_2 \in Z_2$ so there are elements $a_2 \in C$, $b_1 \in B_1 \oplus B_2$ such that $x^{-1}a_2x = a_2b_2$ and $x^{-1}b_1x = b_1z_2$. But then $x^{-1}a_2b_1x = (a_2b_1)b$. Hence $x^{-1}a(a_2b_1)^{-1}x = a(a_2b_1)^{-1}$ so that $a(a_2b_1)^{-1} \in Z$; i.e., $a = a_2b_1z_1$ where $a_2 \in C$, $b_1 \in B_1 \oplus B_2$, $z_1 \in Z$. Therefore, $H = Z_1 \oplus Z_2 \oplus B_1 \oplus B_2 \oplus C$. ∎

It is clear from Proposition 4 that, to each $3-3$ group G, we can assign the invariants $\langle e_1, e_2, e_3\rangle$, respectively, the number of basis elements of Z_1, B_1, and C. That is to say, for each ordered triple $\langle e_1, e_2, e_3\rangle$ there is a unique $3-3$ group G which has subgroups Z_1, Z_2, B_1, B_2, C meeting the requirements of Proposition 4 and such that the dimensions of Z_1, B_1, and C are, respectively, e_1, e_2, and e_3. That this results in Theorem 10b is clear.

THEOREM 10b. *Every $3-3$ group G is $\aleph_0$-categorical.*

Proof. Left for the reader.

Combining Theorems 10a, 10b, and 5 we obtain the following result.

THEOREM 10. *Every $p-3$ group G, where p is a prime, is $\aleph_0$-categorical.*

Combining Theorem 10 with the proof of Theorem 6 we obtain the following result.

THEOREM 11. *Every $n - 3$ group G, where n is square-free, is $\aleph_0$-categorical.*

The methods used thus far in this section can be extended to obtain the same results for every $n - q$ group G, where n is square-free and q is a prime.

THEOREM 12. *Let p and q be primes. Then any $p - q$ group G is $\aleph_0$-categorical.*

Proof. There are three distinct cases: (1) $q \mid p - 1$; (2) $q \nmid p - 1$ and $q \neq p$; (3) $q = p$. For each of these three cases we must provide a group-theoretic analysis of $p - q$ groups and translate this analysis into the first-order predicate calculus. Case 1 has already been dealt with in Proposition 1. Cases 2 and 3 respectively depend on Propositions 3′ and 4′ below, whose proofs are generalizations of the proofs of Propositions 3 and 4 and are therefore omitted. ∎

The following corollary can be obtained by combining Theorem 12 with the proof of Theorem 6.

THEOREM 13. *Let n be square-free and q a prime. Then every $n - q$ group G is $\aleph_0$-categorical.*

PROPOSITION 3′. *Let p and q be primes such that $q \nmid p - 1$ and $q \neq p$. Let G be a $p - q$ group. Then there are subgroups Z_1 and B of H such that:*

$$\begin{aligned}
\text{(i)}\quad Z_1 = \Big\{ z \in H \mid (\exists a_1)(\exists a_2) \cdots (\exists a_{q-1}) \Big(&\bigwedge_i a_i \in H \\
&\wedge \bigwedge_{i<q-1} x^{-1} a_i x = a_i a_{i+1} \\
&\wedge x^{-1} a_{q-1} x = a_1^{-\binom{q}{1}} a_2^{-\binom{q}{2}} \cdots a_{q-2}^{-\binom{q}{q-2}} a_{q-1}^{1-\binom{q}{q-1}} z \Big) \Big\} \\
= \{ z \in H \mid x^{-1} z x = z \};&
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad B = \Big\{ a_1 \in H \mid (\exists a_2) \cdots (\exists a_{q-1}) \Big(&\bigwedge_i a_i \in H \\
&\wedge \bigwedge_{i<q-1} x^{-1} a_i x = a_i a_{i+1} \\
&\wedge x^{-1} a_{q-1} x = a_1^{-\binom{q}{1}} a_2^{-\binom{q}{2}} \cdots a_{q-2}^{-\binom{q}{q-2}} a_{q-1}^{1-\binom{q}{q-1}} \Big) \Big\};
\end{aligned}$$

(iii) $Z_1 \lhd G$, $B \lhd G$;

(iv) $H = Z_1 \oplus B$;

(v) $B = \sum_{i \in I}^{\oplus} B_i$ *where, for each $i \in I$,*

$B_i \lhd G$ and is generated by exactly $q - 1$ elements.

PROPOSITION 4′. *Let G be a $p - p$ group. Then there are subgroups $Z_1^1, Z_2^1, Z_1^2, Z_2^2, \ldots, Z_1^{p-1}, Z_2^{p-1}, C$ of H such that*

(i) $H = Z_1^1 \oplus Z_2^1 \oplus Z_1^2 \oplus Z_2^2 \oplus \cdots \oplus Z_1^{p-1} \oplus Z_2^{p-1} \oplus C$;

(ii) $Z_1^1 \oplus Z_2^1 = \{z \in H \mid x^{-1}zx = z\}$;

(iii)$_i$ $$Z_1^1 \oplus Z_2^1 \oplus \cdots \oplus Z_1^{i+1} \oplus Z_2^{i+1} = \Big\{b_0 \in H \mid (\exists b_1)(\exists b_2) \cdots (\exists b_i) \Big[\bigwedge_{0 \leqslant j < i} x^{-1}b_jx = b_jb_{j+1} \wedge x^{-1}b_ix = b_i\Big]\Big\};$$

(iv) $Z_2^1 = \{z \in H \mid (\exists b)(b \in H \wedge x^{-1}bx = bz) \wedge x^{-1}zx = z\}$;

(v) $Z_2^1 \oplus Z_2^2 \oplus \cdots \oplus Z_2^{p-1} = \{b \in H \mid (\exists a)(x^{-1}ax = ab)\}$;

(vi) *the map g: $C \to Z_2^{p-1}$ defined by $g(a) = b$ if $x^{-1}ax = ab$ is an isomorphism*;

(vii)$_i$ *the map f_i: $Z_1^{i+1} \oplus Z_2^{i+1} \to Z_2^i$ defined by $f_i(b) = z$ if $x^{-1}bx = bz$ is an isomorphism.*

Proof. The proofs of these propositions are generalizations of the proofs of Propositions 3 and 4 and are left to the reader. ∎

Before concluding this section, we will present one result which is particularly useful for certain $n - k$ groups where the hypotheses of the various theorems above do not hold, for example when n is not square-free.

THEOREM 14. *Let G be a group with a normal Abelian subgroup H of finite index. Assume that there is a natural number M such that for each $h \in H$ there are subgroups H_h and H_h^* of H which are normal in G such that $H = H_h \oplus H_h^*$, $h \in H_h$ and $|H_h| \leqslant M$. Then G is $\aleph_0$-categorical.*

Proof. By repeated use of the hypotheses we can write $H = \sum_{i \in N}^{\oplus} H_i$ where $H_i \lhd G$ and $|H_i| \leqslant M$ for each i. Let $K < G$ be such that $G = \bigcup_{x \in K} Hx$, and let $G_i = \bigcup_{x \in K} H_ix$ for each i. Define $G_i \sim G_j$ if there is an isomorphism between them which fixes each of the elements of K. Since $|G_i| \leqslant M \cdot |K|$ for each i, there are only a finite number of equivalence classes.

Let us assume for the moment that there is but one equivalence class. Let $K = \{1, x_1, x_2, \ldots, x_q\}$ and let $\phi(v_1, v_2, \ldots, v_n)$ be a property, in the language obtained by adding $q + 1$ constant symbols $a_0, a_1, \ldots, a_q$, which says that the $n(q + 1)$ elements of the form v_sa_t ($1 \leqslant s \leqslant n$, $0 \leqslant t \leqslant q$) are all different and form a group isomorphic to G_i, where the v_s's form the subgroup H_i and the a_t's form the subgroup K.

Let the set T of statements consist of the axioms of group theory together with the statements (for each m)

$$
\begin{aligned}
&(\exists v_1{}^1) \cdots (\exists v_n{}^1) \cdots (\exists v_1{}^m) \cdots (\exists v_n{}^m) \Big[\bigwedge_{1 \leqslant i \leqslant m} \phi(v_1{}^i, v_2{}^i, \ldots, v_n{}^i) \\
&\quad \wedge \bigwedge_{i,s,i',s'} (v_s{}^i v_{s'}^{i'} = v_{s'}^{i'} v_s{}^i) \\
&\quad \wedge \bigwedge_{i_1, s_1, \ldots, i_k, s_k, t} (v_{s_1}^{i_1} v_{s_2}^{i_2} \cdots v_{s_k}^{i_k} a_t = 1 \Rightarrow v_{s_1}^{i_1} = 1 \wedge v_{s_2}^{i_2} = 1 \\
&\quad \wedge \cdots \wedge v_{s_k}^{i_k} = 1 \wedge a_t = 1) \wedge (\forall w) \Big\{ \bigwedge v_{s_1}^{i_1} v_{s_2}^{i_2} \cdots v_{s_k}^{i_k} a_t \neq w \\
&\quad \Rightarrow (\exists v_1)(\exists v_2) \cdots (\exists v_n) \Big[\phi(v_1, v_2, \ldots, v_n) \\
&\quad \wedge \bigwedge \bigwedge_{s} v_{s_1}^{i_1} v_{s_2}^{i_2} \cdots v_{s_k}^{i_k} a_t \neq v_s \wedge \bigvee v_{s_1}^{i_1} v_{s_2}^{i_2} \cdots v_{s_k}^{i_k} v_s a_t = w \Big] \Big\} \Big].
\end{aligned}
$$

Then, using Lemma 1, we conclude that G is $\aleph_0$-categorical.

If there is more than one equivalence class we use the same technique used in Theorem 4, or more recently in Theorem 8, to construct a set T of statements for the whole group out of the sets of statements for the components. ∎

It should be noted that Theorem 12 does not include Theorem 5; for example, in the group G of Theorem 6, if we take $h = a_2$ then $\{a_2, z_2\}$ form a normal subgroup $H_{a_2} \lhd G$. But the remainder of H, i.e. that generated by the remaining generators, does not form a normal subgroup of G, since $xa_4x = a_4z_2$. This situation is typical, and, in some sense, the point of the proofs of the theorems in this section is to get around such situations.

5. Direct Limits of Finite Groups

In [13] we showed that the group $GL_2(B)$, where B is a countable atomless Boolean ring, is $\aleph_0$-categorical. In this section we shall claim that any group so constructed is also $\aleph_0$-categorical.

Let H be a finite group and for each $n \in N$ let $H^{(n)}$ be the direct sum of 2^n copies of H. For each n define $\sigma_n \colon H^{(n)} \to H^{(n+1)}$ by $(\sigma_n(\alpha))_j = (\alpha)_{[j/2]}$ for each j, $0 \leqslant j < 2^{n+1}$. (If $\alpha \in \sum_{i<n}^{\oplus} G_i$ then $(\alpha)_i$ is the component of α in G_i.) Thus, for example, if $\langle a, b, c, d \rangle \in H^{(2)}$ then

$$\sigma_2(\langle a, b, c, d \rangle) = \langle a, a, b, b, c, c, d, d \rangle \in H^{(3)}.$$

It is clear that each σ_n is a monomorphism from $H^{(n)}$ to $H^{(n+1)}$. Furthermore if for each m and n, with $m \leqslant n$, we define σ_{mn} to be $\sigma_{n-1} \cdots \sigma_{m+1}\sigma_m$ then σ_{mn}

is a monomorphism from $H^{(m)}$ to $H^{(n)}$. Thus $\{H^{(n)} \mid n \in N\}$ together with the monomorphisms $\{\sigma_{mn} \mid m \leqslant n\}$ form a direct system of groups.

Let H^R the direct limit of this direct system. In [13] we showed that if $H = S_3$ then $H^R \simeq GL_2(B)$ where B is a countable atomless Boolean ring. We used this description of H^R (together with the fact that such a ring is $\aleph_0$-categorical, any two countable atomless Boolean rings being isomorphic) to show that $S_3{}^R$ is $\aleph_0$-categorical. But this is true in general.

THEOREM 15. *Let H be any finite group. Then the group H^R is $\aleph_0$-categorical.*

We will not present a proof of this theorem here. We had intended to prove a generalized version of this theorem elsewhere, but have since been informed by Philip Olin that Theorem 15 and its generalization are a consequence of the work of Waskiewicz and Weglorz [18].

6. Burnside Groups

Another class of groups of bounded order, which could provide further examples of $\aleph_0$-categorical groups, is the class of Burnside groups. Let $B(n, r)$ be the Burnside group of exponent n on r generators. The Burnside conjecture for n is that $B(n, r)$ is finite for all r. The Burnside conjecture is known to be true for $n = 3, 4, 6$ (see Hall [6, Chap. 18]) and to be false for all odd $n \geqslant 4381$ (see Novikov and Adjan [11]). For the remaining values of n, it is not known whether the Burnside conjecture is true or false.

We first show that if $B(n, r)$ is infinite then it is not $\aleph_0$-categorical. This is a consequence of the following theorem.

THEOREM 16. *Let G be $\aleph_0$-categorical. Then every finitely generated subgroup of G is finite. Moreover for each k there is an l such that every subgroup of G generated by k elements has at most l elements.*

Proof. Assume that $\{a_1, a_2, ..., a_k\}$ generate an infinite subset of G. Let $w_0, w_1, w_2, ...$ be words in $\{a_1, a_2, ..., a_k\}$ which represent different elements in the group generated by $\{a_1, a_2, ..., a_k\}$. For each i let t_i result from w_i by replacing each occurrence of a_h in w_i by v_h, for $1 \leqslant h \leqslant k$. Let ϕ^i be $v_{k+1} = t_i$ and let d_j be $\langle a_1, a_2, ..., a_k, w_j\rangle$. Then d_j has property ϕ^i if and only if $i = j$. Hence G is not $\aleph_0$-categorical. Hence if G is $\aleph_0$-categorical, every finitely generated subgroup of G is finite.

Suppose that there is a sequence $d_0, d_1, d_2, ...$ of elements of G^k and an increasing sequence $n_0, n_1, n_2, ...$ of natural numbers such that for each j the k elements in d_j generate an n_j-element subgroup of G. Let $w_1{}^j, w_2{}^j, ..., w_{n_j}^j$ be words in $\{a_1{}^j, a_2{}^j, ..., a_k{}^j\}$ (where $d_j = \langle a_1{}^j, a_2{}^j, ..., a_k{}^j\rangle$) which represent the different words in the subgroup of G generated by $\{a_1^j, a_2^j, \ldots, a_k^j\}$ and define $t_1^j, t_2^j, \ldots, t_{n_j}^j$ as in the preceding paragraph. For each i let ϕ^i be $\bigwedge_{1 \leqslant a < b \leqslant n_i} (t_a^i \neq t_b^i) \wedge (\forall x)(\bigvee_{1 \leqslant a \leqslant n_i} (x = t_a^i))$. Then d_j has property ϕ^i if and only if $i = j$, so that G is not $\aleph_0$-categorical, contrary to the hypothesis. ∎

We thus need concern ourselves only with those n for which the Burnside conjecture is true. We define $B(n, \aleph_0)$ to be the Burnside group on $\aleph_0$ generators and ask whether $B(n, \aleph_0)$ is $\aleph_0$-categorical if the Burnside conjecture is true for n. In the case where $n = 2$, the Burnside groups $B(n, r)$ are all Abelian, so that $B(2, \aleph_0)$ is the direct sum of two-element groups which by Theorem 2 is $\aleph_0$-categorical.

The case for $n > 2$ is somewhat different. We shall treat here the case where n is an odd prime. We wish to acknowledge at this point the suggestions and assistance given by Dr. Michael O'Nan and Dr. Richard Larson.

THEOREM 17. *$B(3, \aleph_0)$ is not $\aleph_0$-categorical.*

Proof. Let $G = B(3, \aleph_0)$ be generated by $\{x_i \mid i \in N\}$ and for each j let d_j be $(x_1, x_2)(x_3, x_4) \cdots (x_{2j-1}, x_{2j})$, where (x, y) is the commutator of x and y. We shall show that the d_j's are pairwise automorphically inequivalent, so that G cannot be $\aleph_0$-categorical.

Assume then that $i < j$ and that σ is an automorphism of G such that $\sigma(d_i) = d_j$. But $\sigma(d_i) = (\sigma(x_1), \sigma(x_2))(\sigma(x_3), \sigma(x_4)) \cdots (\sigma(x_{2i-1}), \sigma(x_{2i}))$ so that d_j is expressed as a product of fewer than j commutators. It suffices to show that this cannot happen. We shall show that, even modulo G'', d_j cannot be expressed as a product of fewer than j commutators.

Assume then that $d_j = (w_1, w_2)(w_3, w_4) \cdots (w_{2i-1}, w_{2i})$ modulo G''. We may assume that each w_t is a word in $x_1, x_2, \ldots, x_{2j}$ for otherwise we can pass homomorphically to the group generated by $x_1, \ldots, x_{2j}$ and get an expression for d_j in which only $x_1, \ldots, x_{2j}$ occurs.

Since we are working modulo G'' we can assume that the w's are taken modulo G', i.e., we can write

$$w_{2t-1} = x_1^{\alpha_1^{(t)}} \cdots x_{2j}^{\alpha_{2j}^{(t)}} \qquad \text{for} \quad 1 \leqslant t \leqslant i$$

and

$$w_{2t} = x_{2t}^{\beta_1^{(t)}} \cdots x_{2j}^{\beta_{2j}^{(t)}} \qquad \text{for} \quad 1 \leqslant t \leqslant i.$$

Since modulo G'' $(xy, z) = (x, z)(y, z)$ and $(x, yz) = (x, y)(x, z)$ and since $(x, y) = (y, x)^{-1}$ it follows that

$$(w_{2t-1}, w_{2t}) = \prod_{a<b} (x_a, x_b)^{\alpha_a^{(t)}\beta_b^{(t)} - \alpha_b^{(t)}\beta_a^{(t)}}.$$

Now since each element of G'/G'' can be expressed uniquely in the form $\prod_{a<b}(x_a, x_b)^{\epsilon_{ab}}$ (see Hall [6, p. 323]) it follows that the system of $j(2j-1)$ equations below in $(2j)(2i)$ unknowns can be solved (modulo 3) for $\{\alpha_a^{(t)}, \beta_a^{(t)} \mid 1 \leqslant t \leqslant i, 1 \leqslant a \leqslant 2j\}$.

$$E_{ab}\colon \sum_{t=1}^{i} (\alpha_a^{(t)}\beta_b^{(t)} - \alpha_b^{(t)}\beta_a^{(t)}) = \begin{cases} 1 & \text{if } (\exists t)(b = a+1 = 2t) \\ 0 & \text{otherwise} \end{cases}$$

Let V_j be the vector space over Z_3 with basis $v_1, v_2, \ldots, v_{2j}$. Let

$$v^{(t)} = \sum_{a=1}^{2j} \alpha_a^{(t)} v_a \qquad \text{for each } t, \qquad 1 \leqslant t \leqslant i$$

and let $w^{(t)} = \sum_{a=1}^{2j} \beta_a^{(t)} v_a$ for each t, $1 \leqslant t \leqslant i$.

Then

$$(v^{(t)} \otimes w^{(t)}) - (w^{(t)} \otimes v^{(t)}) = \sum_{a,b} (\alpha_a^{(t)}\beta_b^{(t)} - \beta_a^{(t)}\alpha_b^{(t)})(v_a \otimes v_b)$$

so that

$$\begin{aligned} &\sum_{t=1}^{i} [(v^{(t)} \otimes w^{(t)}) - (w^{(t)} \otimes v^{(t)})] \\ &\qquad = \sum_{a,b} \sum_{t=1}^{i} (\alpha_a^{(t)}\beta_b^{(t)} - \beta_a^{(t)}\alpha_b^{(t)})(v_a \otimes v_b) \\ &\qquad = \sum_{a,b} c_{ab}(v_a \otimes v_b). \end{aligned}$$

But if $a < b$ then

$$c_{ab} = \begin{cases} 1 & \text{if } (\exists t)(b = a+1 = 2t) \\ 0 & \text{otherwise,} \end{cases}$$

and if $a > b$ then $c_{ab} = -c_{ba}$, whereas for $a = b$, $c_{ab} = 0$ (all of course modulo 3). Hence

$$\sum_{t=1}^{i} [(v^{(t)} \otimes w^{(t)}) - (w^{(t)} \otimes v^{(t)})] = \sum_{t=1}^{j} [(v_{2t-1} \otimes v_{2t}) - (v_{2t} \otimes v_{2t-1})].$$

But in $V_j \otimes V_j$ the second element has rank $2j$ (since the $2j$ tensors are linearly independent in $V_j \otimes V_j$) and so cannot be expressed as a sum of $2i < 2j$ elements since that must have rank $\leqslant 2i$. This is a contradiction, and the theorem is proven. ∎

It should be noted that the only information used about the group G is that the factor groups be vector spaces (which happens if we replace 3 by an arbitrary prime p) and that G'/G'' be freely generated by $\{(x_i, x_j) \mid i < j\}$. But if a product of these commutators is 1, we can, by choosing a suitable homomorphism, show that each of these commutators is 1. Hence to show that G'/G'' is freely generated by $\{(x_i, x_j) \mid i < j\}$ it suffices to show that in the group $B(p, 2)$ the commutator $(x_1, x_2) \neq 1$, i.e., that $B(p, 2)$ is not Abelian.

But each element of $B(p, 2)$ can be written in the form $x_1^{\epsilon_1} x_2^{\epsilon_2} (x_1, x_2)^{\epsilon_3} z$ where $z \in B(p, 2)''$, so it is sufficient to know that $B(p, 2)/B(p, 2)''$ has p^3 elements. But this latter group can be considered as the semidirect product of $H = \{x_1\}$ and $K = \{x_2, (x_1, x_2)\}$ (subject to the relations $x_1^{-1} x_2 x_1 = x_2 (x_1, x_2)^{-1}$ and $x_1^{-1} (x_1, x_2) x_1 = (x_1, x_2)$) which certainly has more than p^2 and hence at least p^3 elements (see Hall [6, p. 88]). Thus we have proved the following.

THEOREM 18. *$B(p, \aleph_0)$ is not $\aleph_0$-categorical for any odd prime p.*

7. FURTHER REMARKS

In the five sections above we have considered various classes of groups which could contain $\aleph_0$-categorical groups. This investigation is of course a prerequisite to presenting an algebraic characterization of $\aleph_0$-categorical groups. We expect to present further information about this class of groups in subsequent publications.

The reader should observe that for each group shown not to be $\aleph_0$-categorical, there must be a countable group, not isomorphic to it, which is indistinguishable from it in terms of first-order properties. We have not exhibited these groups, and the reader may find it instructive to find them and compare them with the original groups.

The reader who is interested in further study of $\aleph_0$-categorical structures can refer to [1, 14, 18, and 5].

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